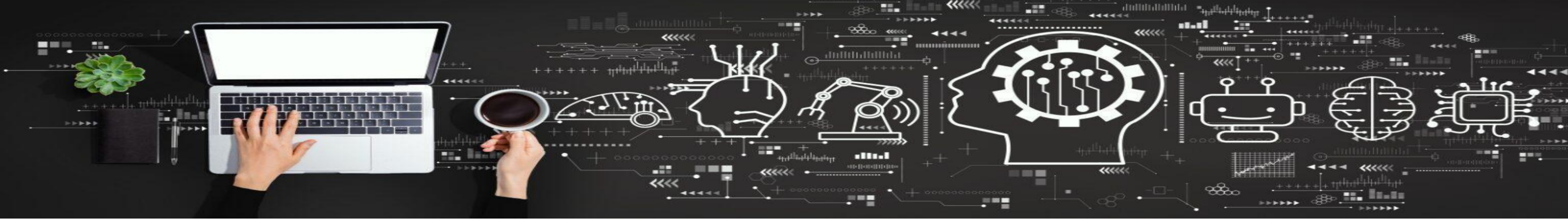

Numerical Methods

Chapter 5:
Eigenvalues and Eigenvectors

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Eigenvalues and vectors:

Let A be a square matrix of order n. λ is an eigenvalue of A if there exists a non-zero vector X such that:

$$AX = \lambda X$$

λ is called the eigenvalue associated with the vector X.

X is called the eigenvector associated with the value λ.

Eigenvalues and vectors:

Let A be a square matrix of order n.λis an eigenvalue of A if there exists a non-zero vector X such that:

- ***The eigenvalues of a diagonal matrix are the elements of the diagonal.***

- ***The sum of the eigenvalues equals the matrix trace.***

- ***The product of the eigenvalues equals the matrix determinant.***

- ***The characteristic polynomial is written:***

$X_{HAS}(x) = (-1)^n X^n + (-1)^{n-1} trace(A) X^{n-1} + \dots + det(HAS)$

Eigenvalues and vectors:

Example :

$$\mathcal{A} = \begin{pmatrix} 1 & 3 & 3 \\ -2 & 11 & -2 \\ 8 & -7 & 6 \end{pmatrix} \qquad \mathcal{X} = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$$

Check that X is an eigenvector of the matrix A associated with an eigenvalueλwhich must be calculated.

Eigenvalues and vectors:

Calculation of eigenvalues:

λ is an eigenvalue of the matrix A; thus there exists a non-zero vector X such that:

$$AX = \lambda X$$

That is to say $(A - \lambda I_n)X = 0$ $\Longrightarrow$ $\text{Det}(A - \lambda I_n) = 0$

Eigenvalues and vectors:

Calculation of eigenvalues:

Example : Let A be the previous square matrix:

$$\mathrm{Det}(A - \lambda I_n) = \left| \begin{pmatrix} 1 & 3 & 3 \\ -2 & 11 & -2 \\ 8 & -7 & 6 \end{pmatrix} - \lambda \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \right| = 0$$

That's to say

$$\begin{vmatrix} 1-\lambda & 3 & 3 \\ -2 & 11-\lambda & -2 \\ 8 & -7 & 6-\lambda \end{vmatrix} = 0$$

Eigenvalues and vectors:

Calculation of eigenvalues:

Example : Let A be the previous square matrix:

$$\begin{vmatrix} 1-\lambda & 3 & 3 \\ -2 & 11-\lambda & -2 \\ 8 & -7 & 6-\lambda \end{vmatrix} = 0$$

$$-\lambda^3 + 18\lambda^2 - 51\lambda - 182 = 0$$

$$-(\lambda + 2)(\lambda - 7)(\lambda - 13) = 0$$

The eigenvalues of A are: $\lambda_1 = -2$; $\lambda_2 = 7$; $\lambda_3 = 13$.

The polynomial characteristic of A is: $X_A(x) = -x^3 + 18x^2 - 51x - 182$

We can verify that: $X_A(x) = (-1)^3x^3 + (-1)^2\mathrm{Trace}(A)x^2 - 51x + \mathrm{Det}(A)$

Calculating the power of a matrix:

Example 1:

Let A be the matrix below; calculate A^n for $0 \leq n \leq 5$.

$$A = \begin{pmatrix} 4 & 1 \\ -2 & 1 \end{pmatrix}$$

Solution :

$$A^0 = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \quad A^1 = \begin{pmatrix} 4 & 1 \\ -2 & 1 \end{pmatrix} \quad A^2 = \begin{pmatrix} 14 & 5 \\ -10 & -1 \end{pmatrix} \quad A^3 = \begin{pmatrix} 46 & 19 \\ -38 & -11 \end{pmatrix}$$

$$A^4 = \begin{pmatrix} 146 & 65 \\ -130 & -49 \end{pmatrix} \qquad A^5 = \begin{pmatrix} 454 & 211 \\ -422 & -179 \end{pmatrix}$$

Calculating the power of a matrix:

Example 2:

Let M be the matrix below; calculate M^2 and M^3.

$$M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

Solution :

$$M^2 = \begin{pmatrix} a^2 + bc & ab + db \\ ac + dc & d^2 + bc \end{pmatrix}$$

$$M^3 = \begin{pmatrix} a^3 + 2abc + bcd & ba^2 + abd + bd^2 + b^2c \\ ca^2 + acd + bc^2 + cd^2 & d^3 + 2bcd + abc \end{pmatrix}$$

What is complicated in calculating the powers of a matrix is that all the coefficients are dispersed during the multiplications.

Calculating the power of a matrix:

Simple case:

If A is a diagonal matrix, its power A^n is diagonal too and each element a_{ii} from A^n equal to the element a_{ii} of A raised to the power n ($a_{ii}^{(n)}=as_{ii}^{n}$).

Example :

$$M = \begin{pmatrix} 3 & 0 & 0 \\ 0 & 7 & 0 \\ 0 & 0 & 5 \end{pmatrix} \qquad M^n = \begin{pmatrix} 3^n & 0 & 0 \\ 0 & 7^n & 0 \\ 0 & 0 & 5^n \end{pmatrix}$$

Calculating the power of a matrix:

Trick :

If we can factorize A into $A = PDP^{-1}$ suchthat D is a diagonal matrix and P^{-1} is the inverse matrix of P, we can thus easily calculate A^n.

$$A = PDP^{-1}$$

$$A^2 = (PDP^{-1})(PDP^{-1}) = PD(P^{-1}P)DP^{-1} = PDIDP^{-1} = PD^2P^{-1}$$

$$A^3 = (PD^2P^{-1})(PDP^{-1}) = PD^3P^{-1}$$

$$\cdots$$

$$A^n = PD^nP^{-1}$$

Calculating the power of a matrix:

Previous example:

$$A = \begin{pmatrix} 4 & 1 \\ -2 & 1 \end{pmatrix} = \begin{pmatrix} 1 & -1 \\ -1 & 2 \end{pmatrix} \begin{pmatrix} 3 & 0 \\ 0 & 2 \end{pmatrix} \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$$

$$P = \begin{pmatrix} 1 & -1 \\ -1 & 2 \end{pmatrix} \qquad P^{-1} = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix} \qquad D = \begin{pmatrix} 3 & 0 \\ 0 & 2 \end{pmatrix}$$

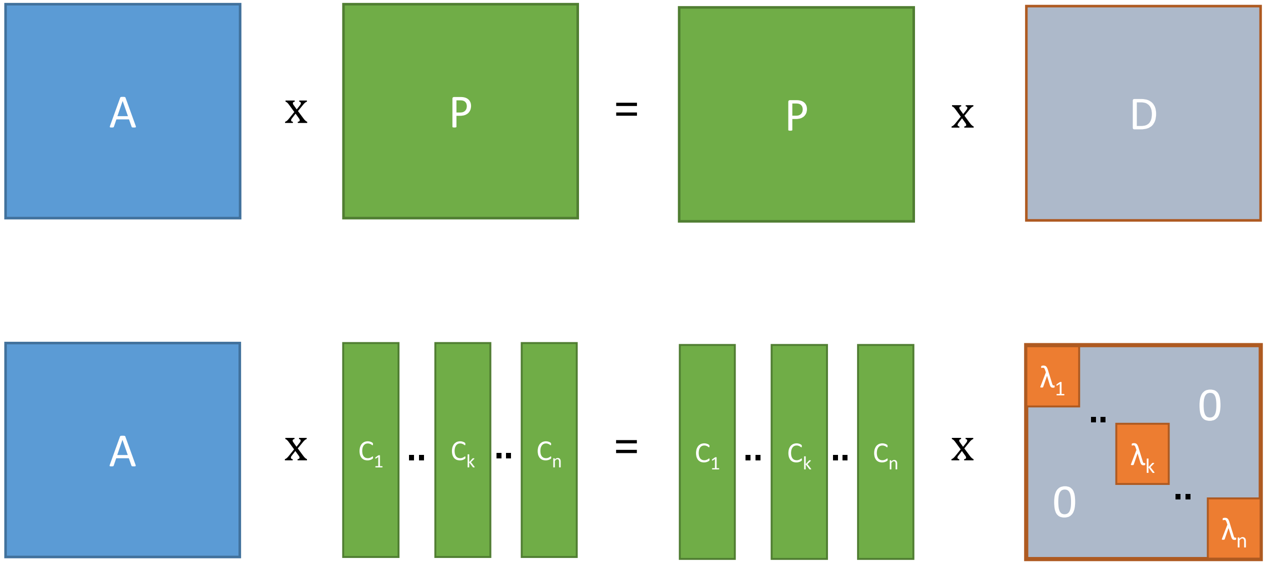
$$A^n = \begin{pmatrix} 1 & -1 \\ -1 & 2 \end{pmatrix} \begin{pmatrix} 3^n & 0 \\ 0 & 2^n \end{pmatrix} \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$$

$$A^n = \begin{pmatrix} -2^n + 2 \times 3^n & -2^n + 3^n \\ 2^{n+1} - 2 \times 3^n & 2^{n+1} - 3^n \end{pmatrix}$$

Calculating the power of a matrix:

Relationship with eigenvalues:

$A = PDP^{-1} \Leftrightarrow AP = PD$

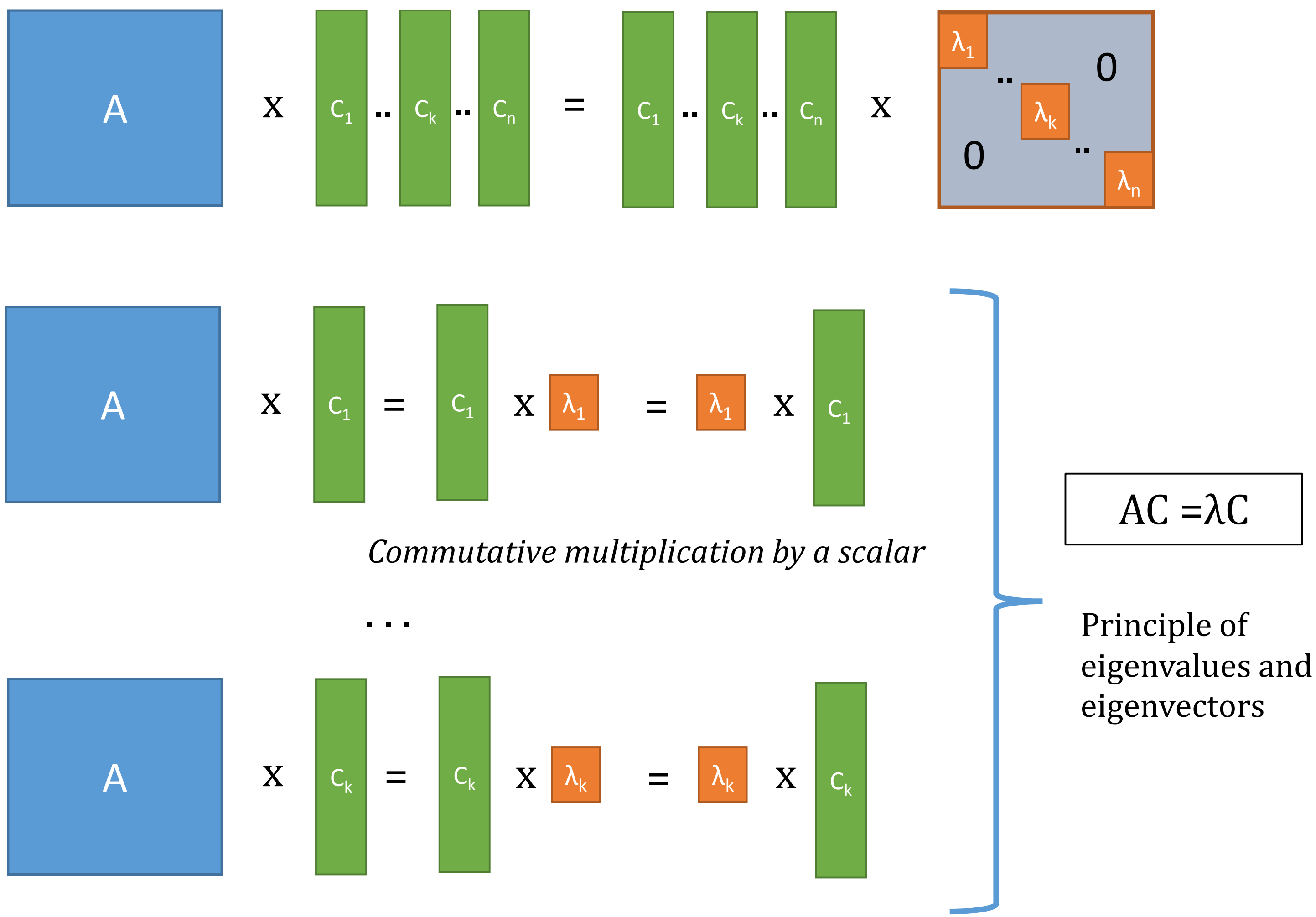


C_i are the columns of P

D is a diagonal matrix

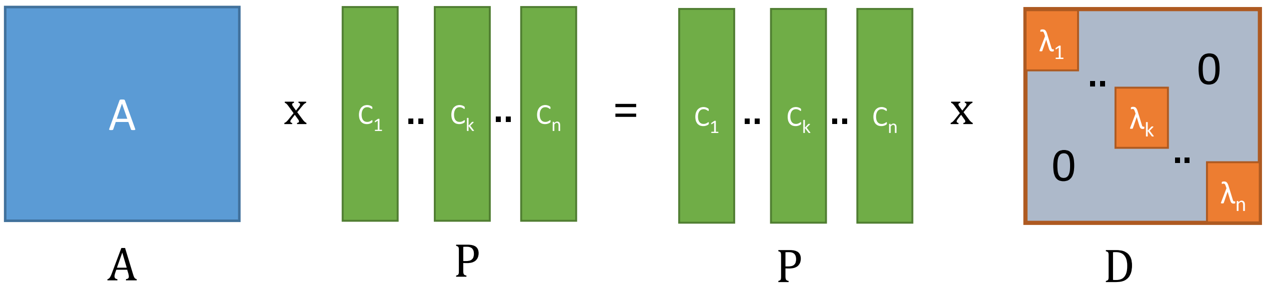
Calculating the power of a matrix:

Relationship with eigenvalues:



Calculating the power of a matrix:

Relationship with eigenvalues:



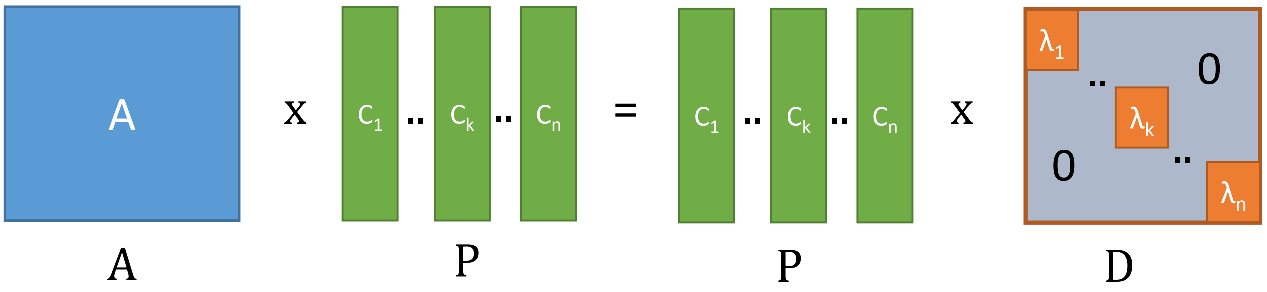
$$AC_k=\lambda_k C_k$$

- The matrix P is formed from the eigenvectors of the matrix A (each column P_k of the matrix P corresponds to an eigenvector C_k).

- The matrix D is formed from the eigenvalues of the matrix A (each element D_k in the diagonal of D corresponds to an eigenvalue λ_k).

Calculating the power of a matrix:

Relationship with eigenvalues:



To calculate A^m, we first calculate the eigenvalues λ_i of A and the corresponding eigenvectors C_i, then we form the matrix D (diagonal matrix) from the eigenvalues λ_i and the matrix P from the eigenvectors C_i from A.
Then we calculate D^m(diagonal matrix containing $(\lambda_1)^n$, $(\lambda_2)^n$..$(\lambda_n)^m$as diagonal values.
Finally we calculate $A^m = P \times D^m \times P^{-1}$.

The order of correspondence between the λ_i and the C_i.

Calculating the power of a matrix:

Example :

Let the matrix A be the following:

$$A = \begin{pmatrix} 3 & 0 & -2 \\ 4 & 5 & -1 \\ 0 & 0 & 7 \end{pmatrix}$$

1- Calculate the eigenvalues and eigenvectors of A, and deduce the matrices P and D such that $A=PDP^{-1}$.

2- Calculate A^5, then find the formula for A^n.

Calculating the power of a matrix:

Example :

1- Eigenvalues:

$$|A-\lambda I| = 0 \Leftrightarrow \det \begin{pmatrix} 3-\lambda & 0 & -2 \\ 4 & 5-\lambda & -1 \\ 0 & 0 & 7-\lambda \end{pmatrix} = 0$$

$$(3-\lambda)(5-\lambda)(7-\lambda) = 0 \Leftrightarrow \lambda_1 = 3; \lambda_2 = 5; \lambda_3 = 7.$$

Calculating the power of a matrix:

Example :

1- Eigenvectors:

➤ **$\lambda_1 = 3$:**

$$X_1 = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \text{ such as } X_1 \neq 0$$

$HASX_1 = \lambda_1 X_1 \Leftrightarrow AX_1 = 3X_1$

$$\begin{cases} 3x_1 - 2x_3 = 3x_1 \\ 4x_1 + 5x_2 - x_3 = 3x_2 \\ 7x_3 = 3x_3 \end{cases}$$

$$\begin{cases} 3x_1 - 2x_3 = 3x_3 \\ 4x_1 + 5x_2 - x_3 = 3x_2 \\ 7x_3 = 3x_3 \Leftrightarrow x_3 = 0 \end{cases} \quad \begin{cases} 3x_1 = 3x_3 \\ x_2 = -2x_1 \\ x_3 = 0 \end{cases} \Leftrightarrow \quad X_1 = \begin{pmatrix} ½ \\ -1 \\ 0 \end{pmatrix}$$

Calculating the power of a matrix:

Example :

1- Eigenvectors:

➢ **$\lambda_2 = 5$:**

$X_2 = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$ such as $X_2 \neq 0$

HAS$X_2 = \lambda_2 X_2 \Leftrightarrow AX_2 = 5X_2$

$$\begin{cases} 3x_1 - 2x_3 = 5x_1 \\ 4x_1 + 5x_2 - x_3 = 5x_2 \\ 7x_3 = 5x_3 \end{cases}$$

$$\begin{cases} 3x_1 - 2x_3 = 5x_1 \\ 4x_1 + 5x_2 - x_3 = 5x_2 \\ 7x_3 = 5x_3 \Leftrightarrow x_3 = 0 \end{cases} \quad \begin{cases} 3x_1 = 5x_1 \Leftrightarrow x_1 = 0 \\ 5x_2 = 5x_2 \\ x_3 = 0 \end{cases} \Leftrightarrow X_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$$

Calculating the power of a matrix:

Example :

1- Eigenvectors:

➢ $\lambda_3 = 7$:

$X_3 = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$ such as $X_3 \neq 0$

$$HASX_3 = \lambda_3 X_3 \Leftrightarrow AX_3 = 7X_3$$

$$\begin{cases} 3x_1 - 2x_3 = 7x_1 \\ 4x_1 + 5x_2 - x_3 = 7x_2 \\ 7x_3 = 7x_3 \end{cases}$$

$$\begin{cases} 3x_1 - 2x_3 = 7x_1 \\ 4x_1 + 5x_2 - x_3 = 7x_2 \\ 7x_3 = 7x_3 \end{cases} \quad \begin{cases} 4x_1 = -2x_3 \Leftrightarrow x_1 = -\frac{1}{2}x_3 \\ x_2 = -3/2x_3 \\ 7x_3 = 7x_3 \end{cases} \Leftrightarrow X_3 = \begin{pmatrix} \frac{1}{2} \\ 3/2 \\ -1 \end{pmatrix}$$

Calculating the power of a matrix:

Example :

1- PDP decomposition^{-1}:

$$D = \begin{pmatrix} 3 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 7 \end{pmatrix}$$

$$P = \begin{pmatrix} ½ & 0 & ½ \\ -1 & 1 & 3/2 \\ 0 & 0 & -1 \end{pmatrix}$$

D is a diagonal matrix containing the eigenvalues of A:

P is a matrix formed from the eigenvectors of A:

Calculating the power of a matrix:

Example :

1- PDP decomposition^{-1}:

$$\lambda_1 \leftrightarrow X_1$$

$$D = \begin{pmatrix} 3 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 7 \end{pmatrix} \qquad P = \begin{pmatrix} \frac{1}{2} & 0 & \frac{1}{2} \\ -1 & 1 & 3/2 \\ 0 & 0 & -1 \end{pmatrix}$$

D is a diagonal matrix containing the eigenvalues of A:

P is a matrix formed from the eigenvectors of A:

The correspondence between the λ_i and X_i

Calculating the power of a matrix:

Example :

1- PDP decomposition^{-1}:

$\lambda_2 \leftrightarrow X_2$

$$D = \begin{pmatrix} 3 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 7 \end{pmatrix} \qquad P = \begin{pmatrix} ½ & 0 & ½ \\ -1 & 1 & 3/2 \\ 0 & 0 & -1 \end{pmatrix}$$

D is a diagonal matrix containing the eigenvalues of A:

P is a matrix formed from the eigenvectors of A:

The correspondence between the λ_i and X_i

Calculating the power of a matrix:

Example :

1- PDP decomposition^{-1}:

$\lambda_3 \leftrightarrow X_3$

$$D = \begin{pmatrix} 3 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 7 \end{pmatrix} \qquad P = \begin{pmatrix} ½ & 0 & ½ \\ -1 & 1 & 3/2 \\ 0 & 0 & -1 \end{pmatrix}$$

D is a diagonal matrix containing the eigenvalues of A:

P is a matrix formed from the eigenvectors of A:

The correspondence between the λ_i and X_i

Calculating the power of a matrix:

Example :

1- PDP decomposition^{-1}:

$$D = \begin{pmatrix} 3 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 7 \end{pmatrix} \qquad P = \begin{pmatrix} \frac{1}{2} & 0 & \frac{1}{2} \\ -1 & 1 & 3/2 \\ 0 & 0 & -1 \end{pmatrix}$$

$$P^{-1} = \begin{pmatrix} 2 & 0 & 1 \\ 2 & 1 & 5/2 \\ 0 & 0 & -1 \end{pmatrix}$$

Calculating the power of a matrix:

Example :

We can clearly verify that:

$$\underbrace{\begin{pmatrix} 1/2 & 0 & 1/2 \\ -1 & 1 & 3/2 \\ 0 & 0 & -1 \end{pmatrix}}_{P} * \underbrace{\begin{pmatrix} 3 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 7 \end{pmatrix}}_{D} * \underbrace{\begin{pmatrix} 2 & 0 & 1 \\ 2 & 1 & 5/2 \\ 0 & 0 & -1 \end{pmatrix}}_{P^{-1}}$$

$$= \underbrace{\begin{pmatrix} 3 & 0 & -2 \\ 4 & 5 & -1 \\ 0 & 0 & 7 \end{pmatrix}}_{A}$$

Calculating the power of a matrix:

Example :

2- Power of A^5:

$A = P*D*P^{-1} \Longleftrightarrow A^5 = P*D^5*P^{-1}$

$$D^5 = \begin{pmatrix} 3^5 & 0 & 0 \\ 0 & 5^5 & 0 \\ 0 & 0 & 7^5 \end{pmatrix} = \begin{pmatrix} 243 & 0 & 0 \\ 0 & 3125 & 0 \\ 0 & 0 & 16807 \end{pmatrix}$$

Calculating the power of a matrix:

Example :

2- Power of A^5:

$A^5 = P*D^5*P^{-1} =$

$$\begin{pmatrix} \frac{1}{2} & 0 & \frac{1}{2} \\ -1 & 1 & 3/2 \\ 0 & 0 & -1 \end{pmatrix} * \begin{pmatrix} 3^5 & 0 & 0 \\ 0 & 5^5 & 0 \\ 0 & 0 & 7^5 \end{pmatrix} * \begin{pmatrix} 2 & 0 & 1 \\ 2 & 1 & 5/2 \\ 0 & 0 & -1 \end{pmatrix}$$

$$= \begin{pmatrix} 3^5 & 0 & \frac{1}{2}*3^5-\frac{1}{2}*7^5 \\ -2*3^5+2*5^5 & 5^5 & -3^5+5/2*5^5-3/2*7^5 \\ 0 & 0 & 7^5 \end{pmatrix}$$

Calculating the power of a matrix:

Example :

2- Power of A^5:

$A^5 = P * D^5 * P^{-1} =$

$$\begin{pmatrix} \frac{1}{2} & 0 & \frac{1}{2} \\ -1 & 1 & 3/2 \\ 0 & 0 & -1 \end{pmatrix} * \begin{pmatrix} 3^5 & 0 & 0 \\ 0 & 5^5 & 0 \\ 0 & 0 & 7^5 \end{pmatrix} * \begin{pmatrix} 2 & 0 & 1 \\ 2 & 1 & 5/2 \\ 0 & 0 & -1 \end{pmatrix}$$

$$= \begin{pmatrix} 243 & 0 & -8282 \\ 5764 & 3125 & -17641 \\ 0 & 0 & 16807 \end{pmatrix}$$

Calculating the power of a matrix:

Example :

2- Power of A^n:

$A^n = P*D^n*P^{-1} =$

$$\begin{pmatrix} 1/2 & 0 & 1/2 \\ -1 & 1 & 3/2 \\ 0 & 0 & -1 \end{pmatrix} * \begin{pmatrix} 3^n & 0 & 0 \\ 0 & 5^n & 0 \\ 0 & 0 & 7^n \end{pmatrix} * \begin{pmatrix} 2 & 0 & 1 \\ 2 & 1 & 5/2 \\ 0 & 0 & -1 \end{pmatrix}$$

$$= \begin{pmatrix} 3^n & 0 & 1/2*3^n - 1/2*7^n \\ -2*3^n+2*5^n & 5^n & -3^n+5/2*5^n-3/2*7^n \\ 0 & 0 & 7^n \end{pmatrix}$$

Appendix:

Exercise :

Let the matrix A be the following:

$$A = \begin{pmatrix} 2 & 1 & 0 \\ 1 & 3 & 1 \\ 0 & 1 & 2 \end{pmatrix}$$

- Calculate the determinant and trace of A.
- Give the characteristic polynomial of A.
- Knowing that$\lambda=1$ is an eigenvalue of A, without using the formula $|A-\lambda I|=0$ calculate the 2 remaining values.
- Give the PDP decomposition^{-1}from A then deduce the formula of A^n.
- Taking advantage of the formula of A^nalready calculated, deduce $A^{-1}$and A^{-2}.
- Compare the results obtained with the traditional calculation of the inverse of a matrix.