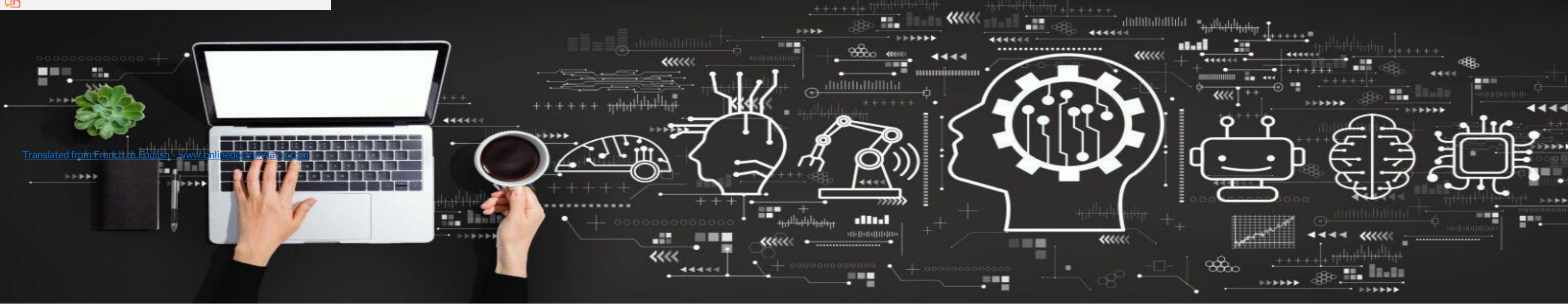

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Numerical Methods

chapter 3:
Solving Linear Systems
Direct Methods

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Solving Linear Systems

Direct methods

- **The Gauss Method**

  LU factorisation

The Gauss method (simple = without exchange)

Linear system

$$
(s)\quad \begin{cases} a11_{X1} + a12_{X2} + .. + a_{1n}Xn = b_1 \\ a21_{X1} + a22_{X2} + .. + a_{2n}Xn = b_2 \\ \quad . \quad . \quad . \\ an1_{X1} + an2_{X2} + .. + ann_{Xn} = b_n \end{cases}
$$

Linear system (square)

$$
\begin{matrix} a_{11} & a_{12} & \ldots & a_{1n} & b_1 \\ a_{21} & a_{22} & \ldots & a_{2n} & b_2 \\ \vdots & & \ldots & & \vdots \\ a_{n1} & a_{n2} & \ldots & a_{nn} & b_n \end{matrix}
$$

Associated matrix (augmented)

1- The Gauss method

Principle of the method :

Transfer the square linear system into an equivalent triangular linear system.

2/3 n^3 + $O(n^2)$ operations

Solve the system (by back substitution)

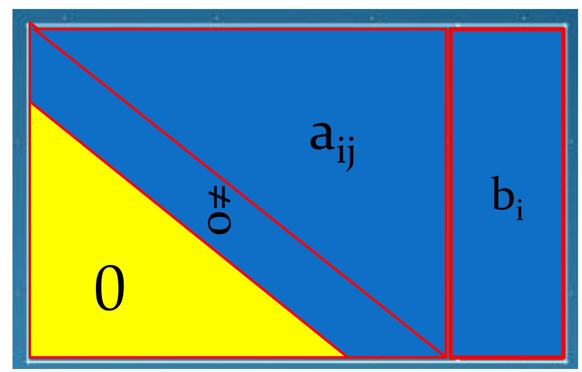
n^2 operations

When n is very large, the number of operations → 2/3 n^3.

$$A = \begin{matrix} a_{11} & a_{12} & \dots & a_{1n} & b_1 \\ a_{21} & a_{22} & \dots & a_{2n} & b_2 \\ \vdots & & \dots & & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nn} & b_n \end{matrix}$$

Associated matrix (augmented)

$A^{(1)} =$



Triangular matrix (upper)

1- The Gauss method

Principle of the method :

Transfer the square linear system into an equivalent triangular linear system.

2/3 n^3 + $O(n^2)$ operations

Solve the system (by back substitution)

n^2 operations

When n is very large, the number of operations → 2/3 n^3.

$$
(s)\quad \begin{cases} a_{11}x_1 + a_{12}x_2 + .. + a_{1nx4} = \boldsymbol{b_1} \\ a_{22}x_2 + .. + a_{2nx4} = \boldsymbol{b_2} \\ ..... \\ a_{nn}x_4 = \boldsymbol{bn} \end{cases}
$$

$$
A = \begin{matrix} a_{11} & a_{12} & ... & a_{1n} & b_1 \\ 0 & a_{22} & ... & a_{2n} & b_2 \\ : & & ... & & : \\ 0 & 0 & ..0.. & a_{nn} & b_n \end{matrix}
$$

Linear system (square)

Associated matrix (augmented)

1- The Gauss method

Algorithms :

Triangulation algorithm :

```
n, i, j, k: entries;  B:real
A augmented matrix(n, n+1)
For k = 1 to n-1 do
    (Choose the pivot and make the
    necessary permutations)
    For i = k+1 to n do
        B = A_ik
        For j = k to n+1 do
            A_ij = A_ij - B * A_kj / A_kk
        End for
    End for
End for
```

Solving algorithm :

```
A matrix (triangulation results)
n, i, j : entries
For i = n to (-1) 1 do
    X_i = A_i,n+1
    For j = i+1 to n do
        Xi   = X_i - Aij* Xj
    End for
    X_i = X_i / A_ii
End for
```

1- The Gauss method

Example: Consider the following linear system:

$$(s)\quad \begin{cases} 2x_1 + x_2 + 2x_3 - x_4 = \mathbf{8} \\ -x_1 - x_2 + x_3 + x_4 = \mathbf{1} \\ 3x_1 - 3x_2 + 2x_3 + x_4 = \mathbf{17} \\ x_1 + 2x_2 - 3x_3 - 2x_4 = \mathbf{-7} \end{cases}$$

Linear system (square)

2	1	2	-1	8
-1	-1	1	1	1
3	-3	2	1	17
1	2	-3	-2	-7

Associated matrix (augmented)

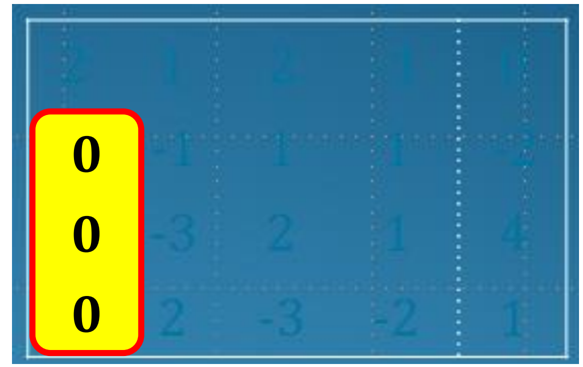
1- The Gauss method

Pivot

$$A = \begin{pmatrix} \mathbf{2} & \mathbf{1} & \mathbf{2} & \mathbf{-1} & \mathbf{8} \\ -1 & -1 & 1 & 1 & 1 \\ 3 & -3 & 2 & 1 & 17 \\ 1 & 2 & -3 & -2 & -7 \end{pmatrix}$$

Associated matrix

$A^{(1)} =$



Triangular matrix

Iteration 1:

- Iteration 1: K = 1 ➔ Pivot = $a_{kk} = a_{11} = 2$.
- The first line remains the same.
- The elements in the 1st column (below the pivot) are null.

1- Gauss method (simple):

Iteration 1 (i=2, j=2):

Pivot

$$A = \begin{pmatrix} 2 & 1 & 2 & -1 & 8 \\ -1 & -1 & 1 & 1 & 1 \\ 3 & -3 & 2 & 1 & 17 \\ 1 & 2 & -3 & -2 & -7 \end{pmatrix}$$

$$= -1 - (-1 \times 1) / 2 = -1/2$$

$$A^{(1)} = \begin{pmatrix} 2 & 1 & 2 & -1 & 8 \\ 0 & -1 & 1 & 1 & 2 \\ 0 & -3 & 2 & 1 & 4 \\ 0 & 2 & -3 & -2 & 1 \end{pmatrix}$$

Calculation rules for other items:

$$a_{ij}^{(1)} = a_{ij} - (a_{ik} \times a_{kj}) / a_{kk}$$

(i = the row, j = the column , k = the iteration)

- $a_{22}^{(1)} = a_{22} - (a_{21} \times a_{12}) / a_{11} = -1 - (-1) \times (1) / 2 = -1/2$

1- Gauss method (simple):

Iteration 1 (i=2, j=3):

$= 1 - (-1 \times 2) / 2 = 2$

Pivot

$$A = \begin{pmatrix} 2 & 1 & 2 & -1 & 8 \\ -1 & -1 & 1 & 1 & 1 \\ 3 & -3 & 2 & 1 & 17 \\ 1 & 2 & -3 & -2 & -7 \end{pmatrix}$$

$$A^{(1)} = \begin{pmatrix} 2 & 1 & 2 & -1 & 8 \\ 0 & -1/2 & 1 & 1 & 2 \\ 0 & -3 & 2 & 1 & 4 \\ 0 & 2 & -3 & -2 & 1 \end{pmatrix}$$

Calculation rules for other items:

$a_{ij}^{(1)} = a_{ij} - (a_{ik} \times a_{kj}) / a_{kk}$ (i = the row, j = the column , k = the iteration)

- $a_{22}^{(1)} = a_{22} - (a_{21} \times a_{12}) / a_{11} = -1 - (-1) \times (1)/2 = -1/2$
- $a_{23}^{(1)} = a_{23} - (a_{21} \times a_{13}) / a_{11} = 1 - (-1) \times (2)/2 = 2$

1- Gauss method (simple):

Iteration 1 (i=2, j=4):

Pivot

$$= 1 - (-1 \times -1) / 2 = \boxed{1/2}$$

$$A = \begin{pmatrix} 2 & 1 & 2 & -1 & 8 \\ -1 & -1 & 1 & 1 & 1 \\ 3 & -3 & 2 & 1 & 17 \\ 1 & 2 & -3 & -2 & -7 \end{pmatrix}$$

$$A^{(1)} = \begin{pmatrix} 2 & 1 & 2 & -1 & 8 \\ 0 & -1/2 & 2 & & \\ 0 & & & & \\ 0 & & & & \end{pmatrix}$$

Calculation rules for other items:

$$a_{ij}^{(1)} = a_{ij} - (a_{ik} \times a_{kj}) / a_{kk}$$

(i = the row, j = the column , k = the iteration)

- $a_{22}^{(1)} = a_{22} - (a_{21} \times a_{12}) / a_{11} = -1 - (-1) \times (1)/2 = -1/2$
- $a_{23}^{(1)} = a_{23} - (a_{21} \times a_{13}) / a_{11} = 1 - (-1) \times (2)/2 = 2$
- $a_{24}^{(1)} = a_{24} - (a_{21} \times a_{14}) / a_{11} = 1 - (-1) \times (-1)/2 = 1/2$

1- Gauss method (simple):

Iteration 1 (i=2, j=5):

Pivot

$$= 1 - (-1 \times 8) / 2 = 5$$

$$A = \begin{pmatrix} 2 & 1 & 2 & -1 & 8 \\ -1 & -1 & 1 & 1 & 1 \\ 3 & -3 & 2 & 1 & 17 \\ 1 & 2 & -3 & -2 & -7 \end{pmatrix}$$

$$A^{(1)} = \begin{pmatrix} 2 & 1 & 2 & -1 & 8 \\ 0 & -1/2 & 2 & 1/2 & \\ 0 & -3 & 2 & 1 & 4 \\ 0 & 2 & -3 & -2 & 1 \end{pmatrix}$$

Calculation rules for other items:

$$a_{ij}^{(1)} = a_{ij} - (a_{ik} \times a_{kj}) / a_{kk}$$

(i = the row, j = the column , k = the iteration)

- $a_{22}^{(1)} = a_{22} - (a_{21} \times a_{12}) / a_{11} = -1 - (-1) \times (1) / 2 = -1/2$
- $a_{23}^{(1)} = a_{23} - (a_{21} \times a_{13}) / a_{11} = 1 - (-1) \times (2) / 2 = 2$
- $a_{24}^{(1)} = a_{24} - (a_{21} \times a_{14}) / a_{11} = 1 - (-1) \times (-1) / 2 = 1/2$
- $a_{25}^{(1)} = a_{25} - (a_{21} \times a_{15}) / a_{11} = 1 - (-1) \times 8/2 = 5$

1- Gauss method (simple):

Iteration 1 (i=3, j=2):

Pivot

$$A = \begin{pmatrix} 2 & 1 & 2 & -1 & 8 \\ -1 & -1 & 1 & 1 & 1 \\ 3 & -3 & 2 & 1 & 17 \\ 1 & 2 & -3 & -2 & -7 \end{pmatrix}$$

$$= -3 - (3 \times 1) / 2 = -9/2$$

$$A^{(1)} = \begin{pmatrix} 2 & 1 & 2 & -1 & 8 \\ 0 & -1/2 & 2 & 1/2 & 5 \\ 0 & -3 & 2 & 1 & 4 \\ 0 & 2 & -3 & -2 & 1 \end{pmatrix}$$

Calculation rules for other items:

$$a_{ij}^{(1)} = a_{ij} - (a_{ik} \times a_{kj}) / a_{kk}$$

(i = the row, j = the column , k = the iteration)

- $a_{32}^{(1)} = a_{32} - (a_{31} \times a_{12}) / a_{11} = -3 - (3) \times (1)/2 = -9/2$

1- Gauss method (simple):

Iteration 1 (i=3, j=3):

Pivot

$$A = \begin{pmatrix} 2 & 1 & 2 & -1 & 8 \\ -1 & -1 & 1 & 1 & 1 \\ 3 & -3 & 2 & 1 & 17 \\ 1 & 2 & -3 & -2 & -7 \end{pmatrix}$$

$$= 2 - (3 \times 2) / 2 = -1$$

$$A^{(1)} = \begin{pmatrix} 2 & 1 & 2 & -1 & 8 \\ 0 & -1/2 & 2 & 1/2 & 5 \\ 0 & -9/2 & 2 & 1 & 4 \\ 0 & 2 & -3 & -2 & 1 \end{pmatrix}$$

Calculation rules for other items:

$$a_{ij}^{(1)} = a_{ij} - (a_{ik} \times a_{kj}) / a_{kk}$$

(i = the row, j = the column , k = the iteration)

- $a_{32}^{(1)} = a_{32} - (a_{31} \times a_{12}) / a_{11} = -3 - (3) \times (1) / 2 = -9/2$
- $a_{33}^{(1)} = a_{33} - (a_{31} \times a_{13}) / a_{11} = 2 - (3) \times (2) / 2 = -1$

1- Gauss method (simple):

Iteration 1 (i=3, j=4):

Pivot

$$A = \begin{matrix} 2 & 1 & 2 & -1 & 8 \\ -1 & -1 & 1 & 1 & 1 \\ 3 & -3 & 2 & 1 & 17 \\ 1 & 2 & -3 & -2 & -7 \end{matrix}$$

$$= 1 - (3 \times -1) / 2 = 5/2$$

$$A^{(1)} = \begin{matrix} 2 & 1 & 2 & -1 & 8 \\ 0 & -1/2 & 2 & 1/2 & 5 \\ 0 & -9/2 & -1 & 1 & 4 \\ 0 & 2 & -3 & -2 & 1 \end{matrix}$$

Calculation rules for other items:

$$a_{ij}^{(1)} = a_{ij} - (a_{ik} \times a_{kj}) / a_{kk}$$

(i = the row, j = the column , k = the iteration)

- $a_{32}^{(1)} = a_{32} - (a_{31} \times a_{12}) / a_{11} = -3 - (3) \times (1)/2 = -9/2$
- $a_{33}^{(1)} = a_{33} - (a_{31} \times a_{13}) / a_{11} = 2 - (3) \times (2)/2 = -1$
- $a_{34}^{(1)} = a_{34} - (a_{31} \times a_{14}) / a_{11} = 1 - (3) \times (-1)/2 = 5/2$

1- Gauss method (simple):

Iteration 1 (i=3, j=5):

$= 17 - (3 \times 8) / 2 = 5$

Pivot

$$A = \begin{pmatrix} 2 & 1 & 2 & -1 & 8 \\ -1 & -1 & 1 & 1 & 1 \\ 3 & -3 & 2 & 1 & 17 \\ 1 & 2 & -3 & -2 & -7 \end{pmatrix}$$

$$A^{(1)} = \begin{pmatrix} 2 & 1 & 2 & -1 & 8 \\ 0 & -1/2 & 2 & 1/2 & 5 \\ 0 & -9/2 & -1 & 5/2 & 4 \\ 0 & 2 & -3 & -2 & 1 \end{pmatrix}$$

Calculation rules for other items:

$$a_{ij}^{(1)} = a_{ij} - (a_{ik} \times a_{kj}) / a_{kk}$$

(i = the row, j = the column , k = the iteration)

- $a_{32}^{(1)} = a_{32} - (a_{31} \times a_{12}) / a_{11} = -3 - (3) \times (1) / 2 = -9/2$
- $a_{33}^{(1)} = a_{33} - (a_{31} \times a_{13}) / a_{11} = 2 - (3) \times (2) / 2 = -1$
- $a_{34}^{(1)} = a_{34} - (a_{31} \times a_{14}) / a_{11} = 1 - (3) \times (-1) / 2 = 5/2$
- $a_{35}^{(1)} = a_{35} - (a_{31} \times a_{15}) / a_{11} = 17 - (3) \times 8 / 2 = 5$

1- Gauss method (simple):

Iteration 1 (i=4, j=2):

$= 2 - (1 \times 1) / 2 = 3/2$

Pivot

$$A = \begin{pmatrix} 2 & 1 & 2 & -1 & 8 \\ -1 & -1 & 1 & 1 & 1 \\ 3 & -3 & 2 & 1 & 17 \\ 1 & 2 & -3 & -2 & -7 \end{pmatrix}$$

$$A^{(1)} = \begin{pmatrix} 2 & 1 & 2 & -1 & 8 \\ 0 & -1/2 & 2 & 1/2 & 5 \\ 0 & -9/2 & -1 & 5/2 & 5 \\ 0 & 2 & -3 & -2 & 1 \end{pmatrix}$$

Calculation rules for other items:

$$a_{ij}^{(1)} = a_{ij} - (a_{ik} \times a_{kj}) / a_{kk}$$

(i = the row, j = the column , k = the iteration)

- $a_{42}^{(1)} = a_{42} - (a_{41} \times a_{12}) / a_{11} = 2 - (1) \times (1)/2 = 3/2$

1- Gauss method (simple):

Iteration 1 (i=4, j=3):

$$= -3 -(1 \times 2) / 2 = -4$$

Pivot

$A =$

2	1	2	-1	8
-1	-1	1	1	1
3	-3	2	1	17
1	2	-3	-2	-7

$A^{(1)} =$

2	1	2	-1	8
0	-1/2	2	1/2	5
0	-9/2	-1	5/2	5
0	3/2	-3	-2	1

Calculation rules for other items:

$$a_{ij}^{(1)} = a_{ij} - (a_{ik} \times a_{kj}) / a_{kk}$$

(i = the row, j = the column , k = the iteration)

- $a_{42}^{(1)} = a_{42} - (a_{41} \times a_{12}) / a_{11} = 2 - (1) \times (1) / 2 = 3/2$
- $a_{43}^{(1)} = a_{43} - (a_{41} \times a_{13}) / a_{11} = -3 - (1) \times (2) / 2 = -4$

1- Gauss method (simple):

Iteration 1 (i=4, j=4):

Pivot

$$= -2 -(1 \times -1) / 2 = -3/2$$

A =

2	1	2	-1	8
-1	-1	1	1	1
3	-3	2	1	17
1	2	-3	-2	-7

$A^{(1)}$=

2	1	2	-1	8
0	-1/2	2	1/2	5
0	-9/2	-1	5/2	5
0	3/2	-4	-2	1

Calculation rules for other items:

$$a_{ij}^{(1)} = a_{ij} - (a_{ik} \times a_{kj}) / a_{kk}$$

(i = the row, j = the column , k = the iteration)

- $a_{42}^{(1)} = a_{42} - (a_{41} \times a_{12}) / a_{11} = 2 - (1) \times (1)/2 = 3/2$
- $a_{43}^{(1)} = a_{43} - (a_{41} \times a_{13}) / a_{11} = -3 - (1) \times (2)/2 = -4$
- $a_{44}^{(1)} = a_{44} - (a_{41} \times a_{14}) / a_{11} = -2 - (1) \times (-1)/2 = -3/2$

1- Gauss method (simple):

Iteration 1 (i=4, j=5):

$= -7 -(1 \times 8) / 2 = -11$

Pivot

$$A = \begin{pmatrix} 2 & 1 & 2 & -1 & 8 \\ -1 & -1 & 1 & 1 & 1 \\ 3 & -3 & 2 & 1 & 17 \\ 1 & 2 & -3 & -2 & -7 \end{pmatrix}$$

$$A^{(1)} = \begin{pmatrix} 2 & 1 & 2 & -1 & 8 \\ 0 & -1/2 & 2 & 1/2 & 5 \\ 0 & -9/2 & -1 & 5/2 & 5 \\ 0 & 3/2 & -4 & -3/2 & \end{pmatrix}$$

Calculation rules for other items:

$$a_{ij}^{(1)} = a_{ij} - (a_{ik} \times a_{kj}) / a_{kk}$$

(i = the row, j = the column , k = the iteration)

- $a_{42}^{(1)} = a_{42} - (a_{41} \times a_{12}) / a_{11} = 2 - (1) \times (1) / 2 = 3/2$
- $a_{43}^{(1)} = a_{43} - (a_{41} \times a_{13}) / a_{11} = -3 - (1) \times (2) / 2 = -4$
- $a_{44}^{(1)} = a_{44} - (a_{41} \times a_{14}) / a_{11} = -2 - (1) \times (-1) / 2 = -3/2$
- $a_{45}^{(1)} = a_{45} - (a_{41} \times a_{15}) / a_{11} = -7 - (1) \times 8 / 2 = -11$

1- Gauss method (simple):

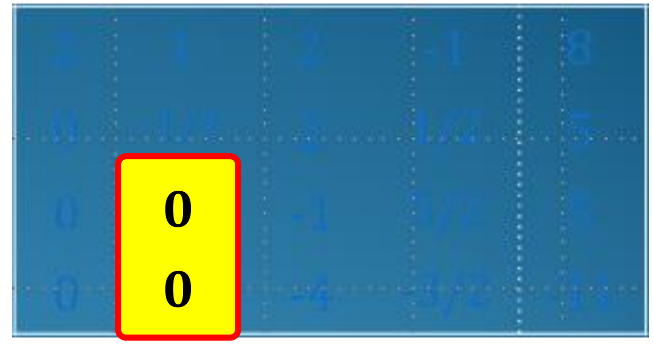
Iteration 2:

A =

2	1	2	-1	8
0	-½	2	½	5
0				
0				

Pivot

$A^{(1)}$=



- Iteration 2: K = 2 ➔ Pivot = a_{kk} = a_{22} = -1/2.

- The 1st religion, the 2nd religion and the 1st column remain the same.

- The elements of the 2nd column (below the pivot) are null.

1- Gauss method (simple):

Iteration 2 (i=3, j=3):

$= -1 \ -(\ {-9/2} \ \times \ 2) \ / \ {-1/2} \ = -19$

Pivot

$A^{(1)} =$

2	1	2	-1	8
0	-1/2	2	1/2	1
0	-9/2	-1	5/2	17
0	3/2	-4	-3/2	-7

$A^{(2)} =$

2	1	2	-1	8
0	-½	2	½	5
0	0			
0	0			

Calculation rules for other items:

$a_{ij}^{(2)} = a_{ij(1)} - (a_{ik(1)} \times a_{kj(1)}) / a_{kk(1)}$ (i = the row, j = the column , k = the iteration)

- $a_{33}^{(2)} = a_{33(1)} - (a_{32(1)} \times a_{23(1)}) / a_{22(1)} = -1 - (-9/2) \times (2) / (-1/2) = -11/2$

1- Gauss method (simple):

Iteration 2 (i=3, j=4):

$= 5/2 - (-9/2 \times ½) / -1/2 = -2$

Pivot

$A^{(1)} =$

2	1	2	-1	0
0	-1/2	2	1/2	-2
0	-9/2	-1	5/2	4
0	3/2	-4	-3/2	1

$A^{(2)} =$

2	1	2	-1	8
0	-½	2	½	5
0	0	-19		
0	0			

Calculation rules for other items:

$a_{ij}^{(2)} = a_{ij(1)} - (a_{ik(1)} \times a_{kj(1)}) / a_{kk(1)}$ (i = the row, j = the column , k = the iteration)

- $a_{33}^{(2)} = a_{33(1)} - (a_{32(1)} \times a_{23(1)}) / a_{22(1)} = -1 - (-9/2) \times (2) / (-1/2) = -11/2$
- $a_{34}^{(2)} = a_{34(1)} - (a_{32(1)} \times a_{24(1)}) / a_{22(1)} = 5/2 - (-9/2) \times (1/2) / (-1/2) = -2$

1- Gauss method (simple):

Iteration 2 (i=3, j=5):

$$= 5 - (-9/2 \times 5) / -1/2 = -40$$

Pivot

$$A^{(1)} = \begin{matrix} 2 & 1 & 2 & -1 & 8 \\ 0 & -1/2 & 2 & 1/2 & 5 \\ 0 & -9/2 & -1 & 5/2 & 5 \\ 0 & 3/2 & -4 & -3/2 & -11 \end{matrix}$$

$$A^{(2)} = \begin{matrix} 2 & 1 & 2 & -1 & 8 \\ 0 & -1/2 & 2 & 1/2 & 5 \\ 0 & 0 & -19 & -2 & \\ 0 & 0 & & & \end{matrix}$$

Calculation rules for other items:

$$a_{ij}^{(2)} = a_{ij(1)} - (a_{ik(1)} \times a_{kj(1)}) / a_{kk(1)}$$

(i = the row, j = the column , k = the iteration)

- $a_{33}^{(2)} = a_{33(1)} - (a_{32(1)} \times a_{23(1)}) / a_{22(1)} = -1 - (-9/2) \times (2) / (-1/2) = -11/2$
- $a_{34}^{(2)} = a_{34(1)} - (a_{32(1)} \times a_{24(1)}) / a_{22(1)} = 5/2 - (-9/2) \times (1/2) / (-1/2) = -2$
- $a_{35}^{(2)} = a_{35(1)} - (a_{32(1)} \times a_{25(1)}) / a_{22(1)} = 5 - (-9/2) \times (5) / (-1/2) = -40$

1- Gauss method (simple):

Iteration 2 (i=4, j=3):

Pivot

$$= -4 - (3/2 \times 2) / -1/2 = 2$$

$A^{(1)} =$

2	1	2	-1	8
0	-1/2	2	1/2	5
0	-9/2	-1	5/2	5
0	3/2	-4	-3/2	-11

$A^{(2)} =$

2	1	2	-1	8
0	-½	2	½	5
0	0	-19	-2	-40
0	0			

Calculation rules for other items:

$$a_{ij}^{(2)} = a_{ij(1)} - (a_{ik(1)} \times a_{kj(1)}) / a_{kk(1)}$$

(i = the row, j = the column , k = the iteration)

- $a_{43}^{(2)} = a_{43(1)} - (a_{42(1)} \times a_{23(1)}) / a_{22(1)} = -4 - (3/2) \times (2) / (-1/2) = 2$

1- Gauss method (simple):

Iteration 2 (i=4, j=4):

Pivot

$$= -3/2 - (3/2 \times ½) / -1/2 = 0$$

$$A^{(1)} = \begin{matrix} 2 & 1 & 2 & -1 & 8 \\ 0 & -1/2 & 2 & 1/2 & 5 \\ 0 & -9/2 & -1 & 5/2 & 5 \\ 0 & 3/2 & -4 & -3/2 & -11 \end{matrix}$$

$A^{(2)}=$

2	1	2	-1	8
0	-½	2	½	5
0	0	-19	-2	-40
0	0	2		

Calculation rules for other items:

$$a_{ij}^{(2)} = a_{ij(1)} - (a_{ik(1)} \times a_{kj(1)}) / a_{kk(1)}$$

(i = the row, j = the column , k = the iteration)

- $a_{43}^{(2)} = a_{43(1)} - (a_{42(1)} \times a_{23(1)}) / a_{22(1)} = -4 - (3/2)x(2)/(-1/2) = 2$
- $a_{44}^{(2)} = a_{44(1)} - (a_{42(1)} \times a_{24(1)}) / a_{22(1)} = -3/2 - (3/2)x(1/2)/(-1/2) = 0$

1- Gauss method (simple):

Iteration 2 (i=4, j=5):

Pivot

$$= -11 - (3/2 \times 5) / -1/2 = 4$$

$A^{(1)} =$

2	1	2	-1	8
0	-1/2	2	1/2	5
0	-9/2	-1	5/2	5
0	3/2	-4	-3/2	-11

$A^{(2)} =$

2	1	2	-1	8
0	-½	2	½	5
0	0	-19	-2	-40
0	0	2	0	

Calculation rules for other items:

$$a_{ij}^{(2)} = a_{ij(1)} - (a_{ik(1)} \times a_{kj(1)}) / a_{kk(1)}$$

(i = the row, j = the column , k = the iteration)

- $a_{43}^{(2)} = a_{43(1)} - (a_{42(1)} \times a_{23(1)}) / a_{22(1)} = -4 - (3/2) \times (2) / (-1/2) = 2$
- $a_{44}^{(2)} = a_{44(1)} - (a_{42(1)} \times a_{24(1)}) / a_{22(1)} = -3/2 - (3/2) \times (1/2) / (-1/2) = 0$
- $a_{45}^{(2)} = a_{45(1)} - (a_{42(1)} \times a_{25(1)}) / a_{22(1)} = -11 - (3/2) \times (5) / (-1/2) = 4$

1- Gauss method (simple):

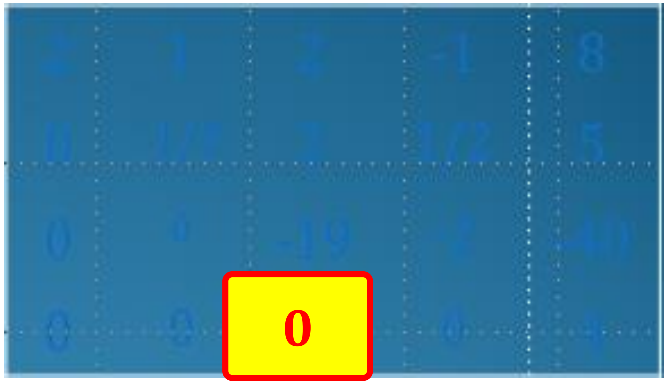
Iteration 3:

$A^{(2)} =$

2	1	2	-1	8
0	-1/2	2	1/2	5
0	0	-19	-2	-40
0	0			

Pivot (−19)

$A^{(3)} =$



- Iteration 3: K = 3 ➔ Pivot = $a_{kk} = a_{33} = -19$.
- The first 3 rows and the first two columns remain the same.
- The elements in the 3rd column (below the pivot) are null.

1- Gauss method (simple):

Iteration 3 (i=4, j=4):

Pivot

$$= 0 - (2 \times -2) / -19 = -4/19$$

$A^{(2)}=$

2	1	2	-1	8
0	-1/2	2	1/2	5
0	0	-19	-2	-40
0	0	2	0	4

$A^{(3)}=$

2	1	2	-1	8
0	-½	2	½	5
0	0	-19	-2	-40
0	0	0		

Calculation rules for other items:

$$a_{ij}^{(2)} = a_{ij(1)} - (a_{ik(1)} \times a_{kj(1)}) / a_{kk(1)}$$

(i = the row, j = the column , k = the iteration)

- ➢ $a_{44}^{(3)} = a_{44(2)} - (a_{43(2)} \times a_{34(1)}) / a_{33(1)} = 0 - (2) \times (-2) / (-19) = -4/19$

1- Gauss method (simple):

Iteration 3 (i=4, j=4):

$$= \quad 4 \; -(\; 2 \; \times \; -40) \; / \; -19 \; = \; -4/19$$

Pivot

$A^{(2)}=$

2	1	2	-1	8
0	-1/2	2	1/2	5
0	0	-19	-2	-40
0	0	2	0	4

$A^{(3)}=$

2	1	2	-1	8
0	-½	2	½	5
0	0	-19	-2	-40
0	0	0	-4/19	

Calculation rules for other items:

$$a_{ij}^{(2)} = a_{ij}^{(1)} - (a_{ik}^{(1)} \times a_{kj}^{(1)}) / a_{kk}^{(1)}$$

(i = the row, j = the column , k = the iteration)

- $a44^{(3)} = a_{44(2)} - (a43_{(2)} \times a34_{(1)})/a33_{(1)} = 0 - (2)\times(-2)/(-19) = -4/19$
- $a45^{(3)} = a_{45(2)} - (a43_{(2)} \times a35_{(1)})/a33_{(1)} = 4 - (2)\times(-40)/(-19) = -4/19$

1- Gauss method (simple):

Final result (triangular matrix):

2	1	2	-1	8
0	-1/2	2	1/2	5
0	0	-19	-2	-40
0	0	0	-4/19	-4/19

1- Gauss method (simple):

System resolution:

2	1	2	-1	8
0	-1/2	2	1/2	5
0	0	-19	-2	-40
0	0	0	-4/19	-4/19

$X_4 = (-4/19)/(-4/19)$	= 1
$X3 = (-40 -(-2X_{4}))/(-19)$	= 2
$X2 = (5 -2X_3-(1/2)X_4)/(-1/2)$	= -1
$X1 = (8 -X2-2X_3+X4)/(2)$	= 3

APPENDICES

Annex 1:

Example: The following linear system:

(s)
$$
\begin{cases}
2x_1 + 3x_2 + 1x_3 = 85 \\
5x_1 + x_2 + 2x = 105 \\
x_1 + 4x_3 = 90
\end{cases}
$$

Linear system (square)

2	3	1	85
5	1	2	105
1	0	4	90

Associated matrix (augmented)

Annex 1:

Iteration1:

Matrice Initiale :

2	3	1	85
5	1	2	105
1	0	4	90

Itération N° 1 :

2	3	1	85
0	-13/2	-1/2	-215/2
0	-3/2	7/2	95/2

Iteration2:

Itération N° 1 :

2	3	1	85
0	-13/2	-1/2	-215/2
0	-3/2	7/2	95/2

Itération N° 2 :

2	3	1	85
0	-13/2	-1/2	-215/2
0	0	47/13	940/13

Solution:

X= 10 15 20

Annex 1:

Example: The following linear system:

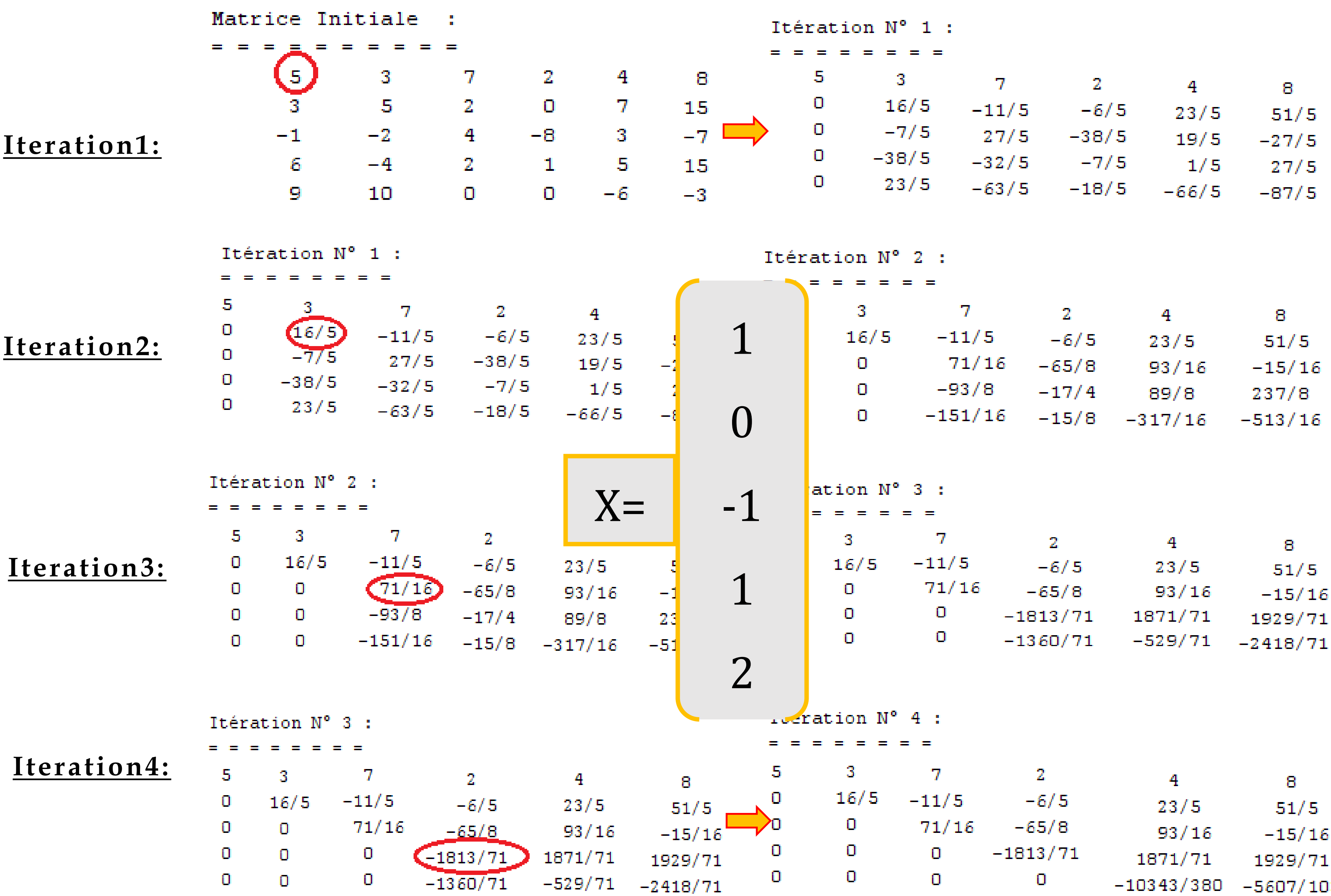
$$
(s)\quad \begin{cases} 5x_1 + 3x_2 + 7x_3 + 2x_4 + 4x_5 = 8 \\ 3x_1 + 5x_2 + 2x_3 + 7x_5 = 15 \\ -x_1 - 2x_2 + 4x_3 - 8x_4 + 3x_5 = -7 \\ 6x_1 - 4x_2 + 2x_3 + x_4 + 5x_5 = 15 \\ 9x_1 + 10x_2 - 6x_5 = -3 \end{cases}
$$

Linear system (square)

5	3	7	2	4	8
3	5	2	0	7	15
-1	-2	4	-8	3	-7
6	-4	2	1	5	15
9	10	0	0	-6	-3

Associated matrix (augmented)

Annex 1:


Iteration1:
Matrice Initiale :
5 3 7 2 4 8
3 5 2 0 7 15
-1 -2 4 -8 3 -7
6 -4 2 1 5 15
9 10 0 0 -6 -3
Itération N° 1 :
5 3 7 2 4 8
0 16/5 -11/5 -6/5 23/5 51/5
0 -7/5 27/5 -38/5 19/5 -27/5
0 -38/5 -32/5 -7/5 1/5 27/5
0 23/5 -63/5 -18/5 -66/5 -87/5
Iteration2:
Itération N° 1 :
Itération N° 2 :
3 7 2 4 8
16/5 -11/5 -6/5 23/5 51/5
0 71/16 -65/8 93/16 -15/16
0 -93/8 -17/4 89/8 237/8
0 -151/16 -15/8 -317/16 -513/16
Iteration3:
Itération N° 2 :
Itération N° 3 :
3 7 2 4 8
16/5 -11/5 -6/5 23/5 51/5
0 71/16 -65/8 93/16 -15/16
0 0 -1813/71 1871/71 1929/71
0 0 -1360/71 -529/71 -2418/71
Iteration4:
Itération N° 3 :
Itération N° 4 :
5 3 7 2 4 8
0 16/5 -11/5 -6/5 23/5 51/5
0 0 71/16 -65/8 93/16 -15/16
0 0 0 -1813/71 1871/71 1929/71
0 0 0 0 -10343/380 -5607/103
X=
1
0
-1
1
2

Annex 2:

The linear system described by the following augmented associated matrix:

1	3	4	2	5	45
2	6	8	2	1	75
3	2	12	5	4	93
1	0	4	-1	3	22
2	1	3	1	1	27

Solve the system by simple Gauss method (without exchange)

What can we notice?

How should this be done?

Pivot Choice Strategies

Pivot Choice Strategies:

1-First Non-Neuel Pivot (NPP):

Used to avoid cases where in a k iteration we arrive at a pivot of zero (Akk= 0).

The principle is to look for the 1ère ligne i in which the element located in the same column k below the pivot zero is not zero (Aik<>0 with i > k).

Thus we make this line (line i) swap with the one containing the zero pivot (line k).

Pivot Choice Strategies:

1-First Non-Neuel Pivot (NPP):

Example:

The linear system described by the following augmented matrix:

1	3	4	2	5	45
2	6	8	2	1	75
3	2	12	5	4	93
1	0	4	-1	3	22
2	1	3	1	1	27

Pivot Choice Strategies:

1-First Non-Neuel Pivot (NPP):

Iteration 1 yields $A^{(1)}$ =

1	3	4	2	5	45
0	0	0	-2	-9	-15
0	-7	0	-1	-11	-42
0	-3	0	-3	-2	-23
0	-5	-5	-3	-9	-63

In iteration 2 we notice that the original pivot is zero (A22 = 0)

1	3	4	2	5	45
0	0	0	-2	-9	-15
0	-7	0	-1	-11	-42
0	-3	0	-3	-2	-23
0	-5	-5	-3	-9	-63

We look for the 1st religion below the pivot whose element in column 2 is not zero. This is line 3.

1	3	4	2	5	45
0	0	0	-2	-9	-15
0	-7	0	-1	-11	-42
0	-3	0	-3	-2	-23
0	-5	-5	-3	-9	-63

Line 2 and line 3 are swapped

1	3	4	2	5	45
0	-7	0	-1	-11	-42
0	0	0	-2	-9	-15
0	-3	0	-3	-2	-23
0	-5	-5	-3	-9	-63

Pivot Choice Strategies:

1-First Non-Neuel Pivot (NPP):

Example:

At iteration 3 we also get an original pivot zero (A33 = 0)

1	3	4	2	5	45
0	-7	0	-1	-11	-42
0	0	0	-2	-9	-15
0	0	0	-18/7	19/7	-5
0	0	-5	-16/7	-8/7	-33

Line 3 must be swapped with line 5

1	3	4	2	5	45
0	-7	0	-1	-11	-42
0	0	-5	-16/7	-8/7	-33
0	0	0	-18/7	19/7	-5
0	0	0	-2	-9	-15

Pivot Choice Strategies:

The two remaining strategies are used to avoid (minimize) the rounding error that can be generated in calculations when there is a large gap between certain elements of the matrix.

2- Best Local Pivot (MPL):

At iteration k, the best element is searched for (which has the highest value by absolute value) in column k. We have to swap the line found with the line k

In iteration 2, the pivot is selected in column 2; The element A42 is thus

Line 2 must be swapped with line 4

5	-1	2	2	16
0	1/5	8/5	-12/5	19/5
0	3/5	-11/5	14/5	-18/5
0	28/5	14/5	24/5	202/5

Pivot Choice Strategies:

3- Best Global Pivot (MPG):

The same principle is applied but on the part of the matrix bounded by the element A_{KK} and the element A_{nn} .

The process can thus lead to a permutation of rows, columns or even rows and columns at the same time.

The permutation of columns k and j automatically causes a permutation in the components Xket $_{\text{Xjof the}}$ solution X vector.

Pivot Choice Strategies:

3- Best Global Pivot (MPG) (example):

In iteration 2, the pivot is chosen in the part delimited by A_{22} and A_{44}

1	5/6	2/3	1/2	25/3
0	5/3	10/3	2	71/3
0	-10/3	-11/3	1	-82/3
0	-8/3	2/3	-4	-2/3

The best item found is A44, which will cause line 2 and 4 to swap; then a swap between columns 2 and 4

1	5/6	2/3	1/2	25/3
0	5/3	10/3	2	71/3
0	-10/3	-11/3	1	-82/3
0	-8/3	2/3	-4	-2/3

Swapping of rows 2 and 4 and columns 2 and 4.

1	1/2	2/3	5/6	25/3
0	-4	2/3	-8/3	-2/3
0	1	-11/3	-10/3	-82/3
0	2	10/3	5/3	71/3

Following the swapping of columns 2 and 4, components X2 and X4 of X are swapped as well.

$$X = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} \rightarrow X = \begin{pmatrix} x_1 \\ x_4 \\ x_3 \\ x_2 \end{pmatrix}$$

Example:

Either the linear system described above by:

1	3	4	2	5	45
2	6	8	2	1	75
3	2	12	5	4	93
1	0	4	-1	3	22
2	1	3	1	1	27

Solve this system by the Gauss method with the pivot choice strategy: PPN, MPL, and MPG.

Example: (Gauss PPN)

Matrice Initiale :

= = = = = = = = = =

1	3	4	2	5	45
2	6	8	2	1	75
3	2	12	5	4	93
1	0	4	-1	3	22
2	1	3	1	1	27

Itération N° 1 :

= = = = = = =

$A^{(1)}$ =

1	3	4	2	5	45
0	0	0	-2	-9	-15
0	-7	0	-1	-11	-42
0	-3	0	-3	-2	-23
0	-5	-5	-3	-9	-63

Itération N° 2 : Pivot A[2,2]=0;

= = = = = = = =

Le prochain élément non nul dans la colonne 2 est A[3,2]

Permutation des lignes 2 et 3

1	3	4	2	5	45
0	-7	0	-1	-11	-42
0	0	0	-2	-9	-15
0	-3	0	-3	-2	-23
0	-5	-5	-3	-9	-63

==> $A^{(2)}$ =

1	3	4	2	5	45
0	-7	0	-1	-11	-42
0	0	0	-2	-9	-15
0	0	0	-18/7	19/7	-5
0	0	-5	-16/7	-8/7	-33

Example: (Gauss PPN)

```
Itération N° 3 :  Pivot A[3,3]=0
= = = = = = = =
 Le prochain élément non nul dans la colonne 3 est A[5,3]
    Permutation des lignes 3 et 5
```

1	3	4	2	5	45
0	-7	0	-1	-11	-42
0	0	-5	-16/7	-8/7	-33
0	0	0	-18/7	19/7	-5
0	0	0	-2	-9	-15

==> $A^{(3)} =$

1	3	4	2	5	45
0	-7	0	-1	-11	-42
0	0	-5	-16/7	-8/7	-33
0	0	0	-18/7	19/7	-5
0	0	0	-2	-9	-15

```
Itération N° 4 :
= = = = = = = =
```

$A^{(4)} =$

1	3	4	2	5	45
0	-7	0	-1	-11	-42
0	0	-5	-16/7	-8/7	-33
0	0	0	-18/7	19/7	-5
0	0	0	0	-100/9	-100/9

```
SOLUTION DU SYSTEME :
= = = = = = = = = =
```

$$X = \begin{pmatrix} 2 \\ 4 \\ 5 \\ 3 \\ 1 \end{pmatrix}$$

Example: (Gauss MPL)

Matrice Initiale :
= = = = = = = = = =

1	3	4	2	5	45
2	6	8	2	1	75
3	2	12	5	4	93
1	0	4	-1	3	22
2	1	3	1	1	27

Itération N° 1 :
= = = = = = = =

Meilleur Pivot dans la colonne 1 est A[3,1] = 3 ==> Permutation des lignes 1 et 3 :

3	2	12	5	4	93
2	6	8	2	1	75
1	3	4	2	5	45
1	0	4	-1	3	22
2	1	3	1	1	27

$A^{(1)} =$

3	2	12	5	4	93
0	14/3	0	-4/3	-5/3	13
0	7/3	0	1/3	11/3	14
0	-2/3	0	-8/3	5/3	-9
0	-1/3	-5	-7/3	-5/3	-35

Itération N° 2 :
= = = = = = = =

$A^{(2)} =$

3	2	12	5	4	93
0	14/3	0	-4/3	-5/3	13
0	0	0	1	9/2	15/2
0	0	0	-20/7	10/7	-50/7
0	0	-5	-17/7	-25/14	-477/14

Example: (Gauss MPL)

Itération N° 3 :
= = = = = = = =

Meilleur Pivot dans la colonne 3 est A[5,3] = -5 ==> Permutation des lignes 3 et 5 :

3	2	12	5	4	93
0	14/3	0	-4/3	-5/3	13
0	0	-5	-17/7	-25/14	-477/14
0	0	0	-20/7	10/7	-50/7
0	0	0	1	9/2	15/2

$A^{(3)} =$

3	2	12	5	4	93
0	14/3	0	-4/3	-5/3	13
0	0	-5	-17/7	-25/14	-477/14
0	0	0	-20/7	10/7	-50/7
0	0	0	1	9/2	15/2

Itération N° 4 :
= = = = = = = =

$A^{(4)} =$

3	2	12	5	4	93
0	14/3	0	-4/3	-5/3	13
0	0	-5	-17/7	-25/14	-477/14
0	0	0	-20/7	10/7	-50/7
0	0	0	0	5	5

SOLUTION DU SYSTEME :
= = = = = = = = = =

$$X = \begin{pmatrix} 2 \\ 4 \\ 5 \\ 3 \\ 1 \end{pmatrix}$$

Example: (Gauss MPG)

```
Matrice Initiale  :
= = = = = = = = = =
     1     3     4     2     5     45
     2     6     8     2     1     75
     3     2    12     5     4     93
     1     0     4    -1     3     22
     2     1     3     1     1     27
```

```
Itération N° 1 :
= = = = = = = =
   Meilleur Pivot dans la matrice est   A[3,3] = 12 :
    ==> Permutation des lignes 1 et 3 :

      3     2    12     5     4     93
      2     6     8     2     1     75
      1     3     4     2     5     45
      1     0     4    -1     3     22
      2     1     3     1     1     27

    ==> Permutation des colonnes 1 et 3
```

12	2	3	5	4	93
8	6	2	2	1	75
4	3	1	2	5	45
4	0	1	-1	3	22
3	1	2	1	1	27

==> (Permutation de X3 et X1) $X = \begin{pmatrix} X_3 \\ X_2 \\ X_1 \\ X_4 \\ X_5 \end{pmatrix}$

$$A^{(1)} = \begin{matrix} 12 & 2 & 3 & 5 & 4 & 93 \\ 0 & 14/3 & 0 & -4/3 & -5/3 & 13 \\ 0 & 7/3 & 0 & 1/3 & 11/3 & 14 \\ 0 & -2/3 & 0 & -8/3 & 5/3 & -9 \\ 0 & 1/2 & 5/4 & -1/4 & 0 & 15/4 \end{matrix}$$

Example: (Gauss MPG)

Itération N° 2 :

= = = = = = = =

$A^{(2)}$ =

12	2	3	5	4	93
0	14/3	0	-4/3	-5/3	13
0	0	0	1	9/2	15/2
0	0	0	-20/7	10/7	-50/7
0	0	5/4	-3/28	5/28	33/14

Itération N° 3 :

= = = = = = = =

Meilleur Pivot dans la matrice est A[3,5] = 4.5

==> Permutation des colonnes 3 et 5

12	2	4	5	3	93
0	14/3	-5/3	-4/3	0	13
0	0	9/2	1	0	15/2
0	0	10/7	-20/7	0	-50/7
0	0	5/28	-3/28	5/4	33/14

==> (Permutation de X5 et X1) : $X = \begin{pmatrix} X_3 \\ X_2 \\ X_5 \\ X_4 \\ X_1 \end{pmatrix}$

$A^{(3)}$ =

12	2	4	5	3	93
0	14/3	-5/3	-4/3	0	13
0	0	9/2	1	0	15/2
0	0	0	-200/63	0	-200/21
0	0	0	-37/252	5/4	173/84

Example: (Gauss MPG)

Itération N° 4 :
= = = = = = = =

$$A^{(4)} = \begin{pmatrix} 12 & 2 & 4 & 5 & 3 & 93 \\ 0 & 14/3 & -5/3 & -4/3 & 0 & 13 \\ 0 & 0 & 9/2 & 1 & 0 & 15/2 \\ 0 & 0 & 0 & -200/63 & 0 & -200/21 \\ 0 & 0 & 0 & 0 & 5/4 & 5/2 \end{pmatrix}$$

Solution Provisoire
= = = = = = = = = =

$$X = \begin{pmatrix} 5 \\ 4 \\ 1 \\ 3 \\ 2 \end{pmatrix}$$

Retablissement de l'ordre
des composantes de X
= = = = = = = = = = = >>

SOLUTION DU SYSTEME :
= = = = == = = = = =

$$X = \begin{pmatrix} 2 \\ 4 \\ 5 \\ 3 \\ 1 \end{pmatrix}$$

End