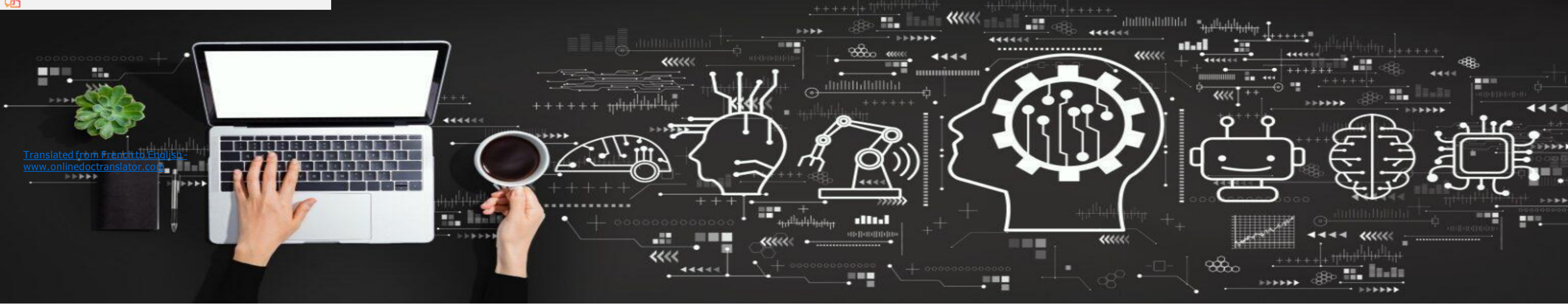

Numerical Methods

Chapter 2: Matrix Calculation

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Course plan

- ***Matrix definitions:***
  - Rectangular matrix; square; row, column.
- ***Special cases of matrix:***
  - Identity matrix; null, triangular (sup/inf); diagonal.
  - Symmetric matrix; trace of a matrix, Transposed matrix ,
- ***Operation on matrices:***
  - Addition, subtraction, multiplication by a scalar, product.
  - Co-matrix (adjoint) , determinant.
  - Inverse matrix (determinant, cofactor).

Definition :

A matrix $A_{m,n}$ is an array of ***mxn*** elements, with ***m*** is the number of lines and ***n*** is the number of columns.

The elements of a matrix $A_{m,n}$ are usually noted by a_{ij}, which ***i*** denotes the line number and ***j*** denotes the column number with ($1 \leq i \leq m$ And $1 \leq j \leq n$).

- A matrix $A_{m,n}$ is said to be of order ***(m,n)*** (or dimension ***mxn***).

- If $m \neq n$, the matrix ***A*** is said to be rectangular, otherwise it is said to be a square matrix.

- If $m = 1$, the matrix ***A*** is a row matrix.

- If $n = 1$, the matrix ***A*** is a column matrix.

Definition :

$$A = \begin{pmatrix} 1 & 0 & -2 & 5 \\ \frac{1}{3} & \sqrt{2} & -1 & 2 \\ 3 & -1 & 0 & -\frac{2}{3} \end{pmatrix}$$

- A is a matrix of order (3, 4) (m=3 rows and n=4 columns, it is noted $\boldsymbol{A_{3.4}}$

- It is rectangular.

Definition :

$$B = \begin{pmatrix} 1 & 0 & \sqrt{3} & 5 \end{pmatrix}$$

$$C = \begin{pmatrix} 1 \\ \frac{1}{3} \\ 3 \end{pmatrix}$$

- B is a row matrix of 4 elements ($\boldsymbol{B_{1.4}}$).

- C is a column matrix of 3 elements ($\boldsymbol{C_{3.1}}$).

Special cases:

- The éléments a_{ii}(below in red) constitute the main diagonal of the matrix.

$$A = \begin{pmatrix} \mathbf{1} & 1 & -2 \\ 2 & \mathbf{-3} & 4 \\ 1 & 0 & \mathbf{0} \end{pmatrix}$$

Special cases:

- If all the éléments a_{ij} of the matrix are zero except the elements of the diagonal, the matrix is called diagonal matrix.

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & 0 \end{pmatrix} \qquad B = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

- If all the éléments a_{ii} of a diagonal matrix are equal to 1, this matrix is called an identity matrix.

Special cases:

- A matrix is said to be upper triangular (resplower) if all elements below (respabove) of the diagonal are zero.

$$A = \begin{pmatrix} 2 & 0 & 0 & 0 \\ \frac{1}{3} & \sqrt{5} & 0 & 0 \\ 1 & -1 & 0 & 0 \\ -5 & 0 & \frac{1}{5} & -\frac{2}{3} \end{pmatrix} \quad B = \begin{pmatrix} 2 & 0 & -2 & \sqrt{3} \\ 0 & \sqrt{5} & -1 & 2 \\ 0 & 0 & 0 & -\frac{2}{3} \\ 0 & 0 & 0 & -\frac{2}{3} \end{pmatrix}$$

Special cases:

- The transposed matrix noted ${}^{T}A$ or A^T of a matrix $A_{(m,n)}$ is a matrix of dimensions nxm, and whose rows are the columns of A and its columns are the rows of A,

$$A = \begin{pmatrix} 2 & -1 & 1 & \sqrt{3} \\ \frac{1}{3} & \sqrt{5} & 0 & 4 \\ 1 & -2 & 0 & \frac{1}{5} \\ 0 & -1 & \frac{3}{8} & -\frac{2}{3} \end{pmatrix} \quad A^T = \begin{pmatrix} 2 & \frac{1}{3} & 1 & 0 \\ -1 & \sqrt{5} & -2 & -1 \\ 1 & 0 & 0 & \frac{3}{8} \\ \sqrt{3} & 4 & \frac{1}{5} & -\frac{2}{3} \end{pmatrix}$$

Special cases:

- A matrix is said to be symmetric if the elements are symmetric about the diagonal (a symmetric matrix is equal to its transposed matrix).

$$A = \begin{pmatrix} 2 & \frac{1}{3} & 1 & \sqrt{3} \\ \frac{1}{3} & \sqrt{5} & 0 & -1 \\ 1 & 0 & 0 & \frac{1}{5} \\ \sqrt{3} & -1 & \frac{1}{5} & -\frac{2}{3} \end{pmatrix}$$

Special cases:

- A zero matrix

$$A = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

- Exercise: Determine x,y,z and t such that: $\begin{pmatrix} x+y & 2z+t \\ x-y & z-t \end{pmatrix} = \begin{pmatrix} 3 & 7 \\ 1 & 5 \end{pmatrix}$

Special cases:

- A square matrix is a matrix whose number of rows is equal to the number of columns.

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \qquad \begin{pmatrix} 1 & 0 & 1 \\ 3 & -1 & 2 \\ 5 & -2 & 0 \end{pmatrix}$$

- The unit matrix is the diagonal matrix that has only 1s in its main diagonal.

$$I_3 \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad XI_n = I_nX = X$$

Operations on matrices:

Addition, subtraction:

- Two matrices **A** = (**aij**) and **B** = (**bij**) of the same type (**n;m**) canbe added or subtracted. The sum (or difference) of these two matrices is a matrix **C** = (**cij**) of the same type such that:

$$\begin{pmatrix} 1 & 0 & 1 \\ 1 & -1 & 2 \end{pmatrix} + \begin{pmatrix} 0 & 1 & 3 \\ -3 & 2 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 4 \\ -2 & 1 & 3 \end{pmatrix}.$$

Multiplication by a scalar :

- All elements of the matrix are multiplied by this scalar.

$$2\begin{pmatrix} 1 & 0 & 1 \\ 3 & -1 & 2 \\ 5 & -2 & 0 \end{pmatrix} = \begin{pmatrix} 2 & 0 & 2 \\ 6 & -2 & 4 \\ 10 & -4 & 0 \end{pmatrix}$$

1. $\alpha (A + B) = \alpha A + \alpha B.$
2. $(\alpha + \beta) A = \alpha A + \beta A.$
3. $\alpha (\beta A) = (\alpha\beta) A.$
4. $IA = A.$

Operations on matrices:

Multiplication of 2 matrices (AxB=C) :

- Multiplication is possible only if the number of columns of A is equal to the number of rows of B. If A(m, p) multiplies B(p, n) then C is a matrix of order (m, n).
- Multiplication is performed by multiplying term by term a row of A with a column of B and adding each of the products.

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{pmatrix} \times \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \\ b_{31} & b_{32} \end{pmatrix} = \begin{pmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{pmatrix}$$

$$\text{où}\{c_{11} = a_{11}\, b_{11} + a_{12}\, b_{21} + a_{13} b_{31}$$
$$\{c_{12} = a_{11}\, b_{12} + a_{12}\, b_{22} + a_{13} b_{32}$$
$$\{c_{21} = a_{21}\, b_{11} + a_{22}\, b_{21} + a_{23} b_{31}$$
$$\{c_{22} = a_{21}\, b_{12} + a_{22}\, b_{22} + a_{23} b_{32}$$

Operations on matrices:

Multiplication of 2 matrices (AxB=C) :

- **Example :** if A is of order 3×3 and B of order 3×2 then A×B is of order 3×2

$$\begin{pmatrix} 1 & 0 & 2 \\ 2 & 1 & -1 \\ 1 & 3 & 2 \end{pmatrix} \times \begin{pmatrix} 2 & 1 \\ 0 & 2 \\ 2 & 2 \end{pmatrix} = \begin{pmatrix} 6 & 5 \\ 2 & 2 \\ 6 & 11 \end{pmatrix}$$

Exercice : **1-** let $A = \begin{pmatrix} 0 & 2 & -2 \\ 6 & -4 & 0 \end{pmatrix}, B = \begin{pmatrix} 2 & 1 & 0 \\ 0 & 1 & 0 \\ 2 & -2 & -3 \end{pmatrix}, C = \begin{pmatrix} 8 & 2 \\ -3 & 2 \\ -5 & 5 \end{pmatrix}, D = \begin{pmatrix} 5 \\ 2 \\ -1 \end{pmatrix}, E = \begin{pmatrix} x & y & z \end{pmatrix}.$

What Products Are Possible? Calculate Them!

2- let $A = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 2 \end{pmatrix} \quad B = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 2 \\ 1 & -1 & 0 \end{pmatrix}.$

Calculate A^2, B^2, AB and BA.

Operations on matrices:

Determinant :

- Let A $\in Mn$ $(\mathbb{R})$ be a square matrix. The determinant of A, denoted det(A), is the element :

- 1) $\det\begin{pmatrix}1 & 2\\-1 & 2\end{pmatrix} = \begin{vmatrix}1 & 2\\-1 & 2\end{vmatrix} = 1.2 - 2(-1) = 4$

- 2)$\det\begin{pmatrix}1 & 1 & 2\\3 & 0 & -1\\1 & 2 & 1\end{pmatrix} = \begin{vmatrix}1 & 1 & 2\\3 & 0 & -1\\1 & 2 & 1\end{vmatrix} = 1\begin{vmatrix}0 & -1\\2 & 1\end{vmatrix}-1\begin{vmatrix}3 & -1\\1 & 1\end{vmatrix}$+2$\begin{vmatrix}3 & 0\\1 & 2\end{vmatrix}$

  - =1(0.1-2(-1))-(3.1-1.(-1))+2(3.2-0.1)=2-4+12=10

Operations on matrices:

Determinant :

- det(AB) = det(A) det(B).
- $det A^t = det A$
- The determinant of any upper(lower) triangular matrix is the product of its diagonal elements.

Operations on matrices:

Inverse matrix:

- Let A be a square matrix of size n*n. If there exists a square matrix B of size n*n such that :

AB=I and BA=I,

we say that A is invertible. We call B the inverse of A and we denote it A^{-1}

$$A^{-p} = (A^{-1})^p = \underbrace{A^{-1}A^{-1}\cdots A^{-1}}_{\text{P factors}}.$$

$$\boxed{(A^{-1})^{-1} = A}$$

$$\boxed{(AB)^{-1} = B^{-1}A^{-1}}$$

Operations on matrices:

Exercice : let $A = \begin{pmatrix} 1 & 2 \\ 0 & 3 \end{pmatrix}$.

Study whether A is invertible?

Operations on matrices:

Solution : It is to study the existence of a matrix $B = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$

with coefficients in IK, such that AB=I and BA=I. Now AB=I is equivalent to:

$$AB = I \iff \begin{pmatrix} 1 & 2 \\ 0 & 3 \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \iff \begin{pmatrix} a+2c & b+2d \\ 3c & 3d \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

This Equality Is Equivalent To The System:

$$\begin{cases} a+2c = 1 \\ b+2d = 0 \\ 3c = 0 \\ 3d = 1 \end{cases} \qquad A^{-1} = \begin{pmatrix} 1 & -\frac{2}{3} \\ 0 & \frac{1}{3} \end{pmatrix}.$$

Operations on matrices:

- If n= 2 :

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

- If $ad - bc \neq 0,$ then A is invertible and

$$A^{-1} = \frac{1}{ad-bc}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

If n>2 , what is the solution ??? → **Solution 1 : with co_matrix**
Solution 2 :Chapter 3(Gauss method)

Operations on matrices:

Practical calculation of the inverse of a matrix :

Is a square matrix of order n, if $det A \neq 0$, then the inverse matrix of A is given by:

- $A^{-1} = \frac{1}{\det A}(Co)^t$

Where the matrix (Co) called the cofactor matrix of A is defined by :

- $Co = (C_{ij}), i, j = 1, .., n \; et \; C_{ij} = (-1)^{i+j}\Delta_{ij}$

Δij being the determinant of the matrix of $Mn{-}1$ $\mathbb{R}$ obtained by removing from A the row i and the column j.

Operations on matrices:

Exemple : Calculate the inverse of the matrix

- $A=\begin{pmatrix} 1 & 1 & 2 \\ 3 & 0 & -1 \\ 1 & 2 & 1 \end{pmatrix}$

Operations on matrices:

Solution : A = 10 ≠ 0 → A is invertible.

- $A^{-1} = \frac{1}{\det A}(Co)^t$

- We first calculate the cofactor matrix.

Operations on matrices:

- $c_{11} = (-1)^{1+1}\Delta_{11} = \begin{vmatrix} 0 & -1 \\ 2 & 1 \end{vmatrix} = 2$; $c_{12} = (-1)^{1+2}\Delta_{12} = -\begin{vmatrix} 3 & -1 \\ 1 & 1 \end{vmatrix} - 4$
- $c_{13} = (-1)^{1+3}\Delta_{13} = \begin{vmatrix} 3 & 0 \\ 1 & 2 \end{vmatrix} = 6$
- $c_{21} = (-1)^{2+1}\Delta_{21} = \begin{vmatrix} 1 & 2 \\ 2 & 1 \end{vmatrix} = 3$; $c_{22} = (-1)^{2+2}\Delta_{22} = \begin{vmatrix} 1 & 2 \\ 1 & 1 \end{vmatrix} = -1$
- $c_{23} = (-1)^{2+3}\Delta_{23} = -\begin{vmatrix} 1 & 1 \\ 1 & 2 \end{vmatrix} = -1$;
- $c_{31} = (-1)^{3+1}\Delta_{31} = \begin{vmatrix} 1 & 2 \\ 0 & -1 \end{vmatrix} = -1$; $c_{32} = (-1)^{3+2}\Delta_{32} = -\begin{vmatrix} 1 & 2 \\ 3 & -1 \end{vmatrix} = 7$
- ; $c_{33} = (-1)^{3+3}\Delta_{33} = \begin{vmatrix} 1 & 1 \\ 3 & 0 \end{vmatrix} = -3.$

Operations on matrices:

- Hence the cofactor matrix is :

$$\begin{pmatrix} 2 & -4 & 6 \\ 3 & -1 & -1 \\ -1 & 7 & -3 \end{pmatrix}$$

- Its transpose is :

$$\begin{pmatrix} 2 & 3 & -1 \\ -4 & -1 & 7 \\ 6 & -1 & -3 \end{pmatrix}$$

Operations on matrices:

- Finally the inverse matrix of A is :

- $A^{-1} = \frac{1}{\det A}(Co)^t = \frac{1}{10}\begin{pmatrix} 2 & 3 & -1 \\ -4 & -1 & 7 \\ 6 & -1 & -3 \end{pmatrix}$

- $A^{-1} = \begin{pmatrix} ^1/_5 & ^3/_{10} & ^{-1}/_{10} \\ ^{-2}/_5 & ^{-1}/_{10} & ^7/_{10} \\ ^3/_5 & ^{-1}/_{10} & ^{-3}/_{10} \end{pmatrix}$

Appendix:

Exercise :

Let the matrix A be the following:

$$A = \begin{pmatrix} 1 & 0 & 2 \\ 0 & -1 & 1 \\ 1 & -2 & 0 \end{pmatrix}$$

- Calculate the determinant and trace of A.
- Calculate $A^3 - A$.
- Deduce that A is invertible then determine A^{-1}.