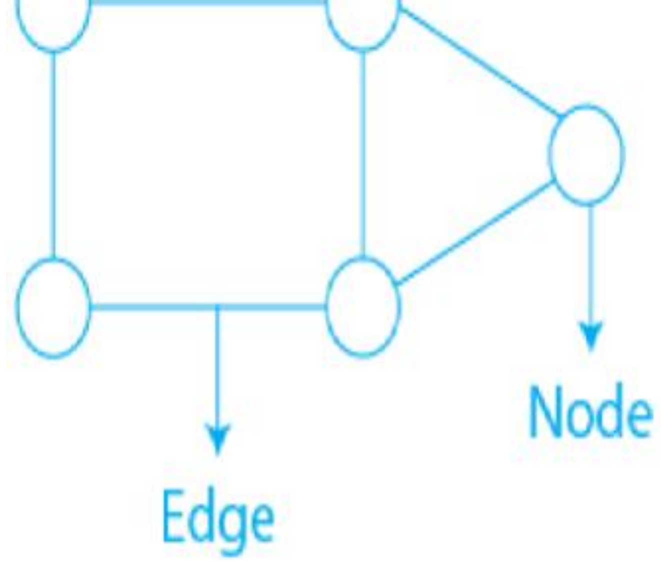
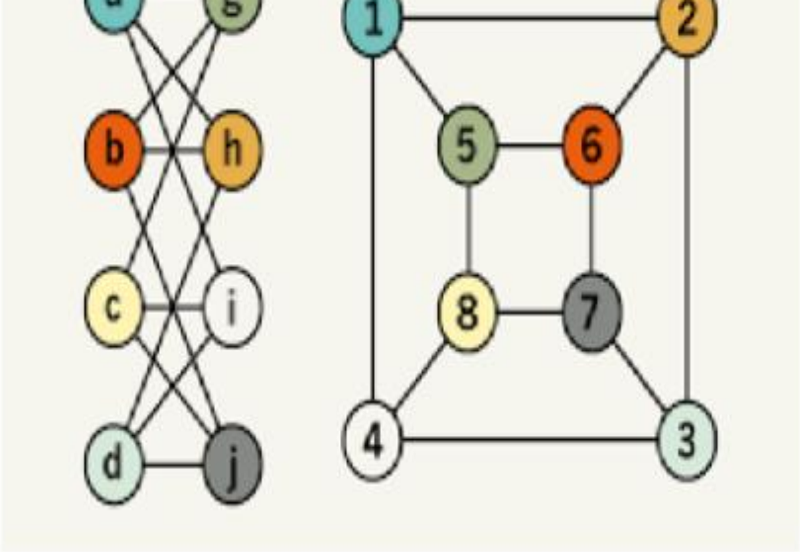
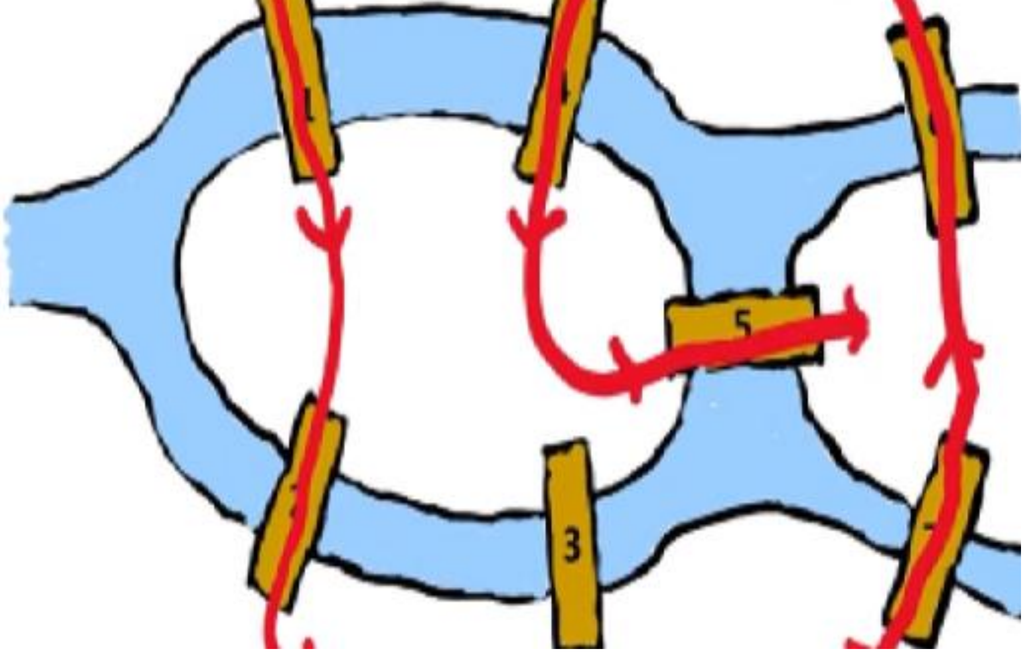






Graph Theory Course

Licence L2

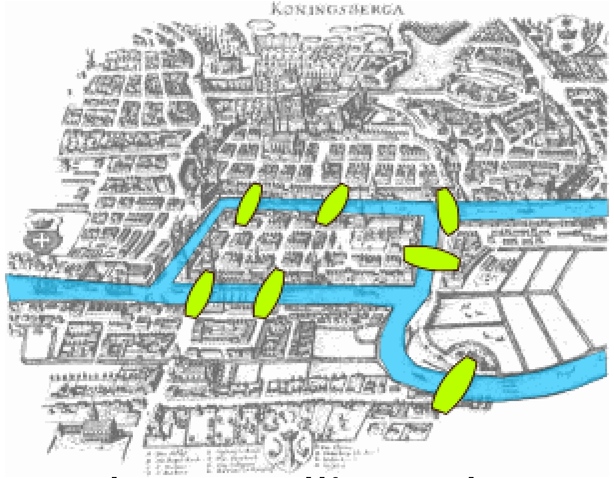
Z.GUELLIL

Chapter 1

Fundamental Concepts

History (1/3)

- Euler (1735) mathematical curiosity: Start from one bank, cross the seven bridges of the city of KÖNESBERG (Germany) once and only once and return to the starting point.



- The Irish mathematician Sir William Hamilton (1805-1865) worked on the travelling salesman problem.
- Francis Guthrie (1852), a South African mathematician, stated the four-colour problem during a discussion with his brother, who asked his teacher Auguste De Morgan if any map could be coloured with four colours so that neighbouring countries had different colours.

History (2/3)

- Julius Petersen at the end of the 19th century was interested in spanning subgraphs, that is, graphs containing all the vertices but only some of the edges. We saw the emergence of graph factorisation problems in this way. A spanning subgraph is called a k-factor if each of its vertices has k edges, and the first theorems were given.
- In the mid-19th century, the British mathematician Arthur Cayley became interested in trees, a special type of graph that does not have a cycle, i.e. in which it is impossible to return to a starting point without going in the opposite direction.

History (3/3)

- But it was only with the Second World War that the practice was organized for the first time and acquired its name.
- In 1940, Patrick Blackett was called by the British general staff to lead the first operational research team to solve certain problems such as the optimal location of surveillance radars.
- From 1946 onwards, TG experienced intense development thanks to researchers motivated by the resolution of concrete problems.
- Among them, Esdger Djikstra (1959) for the routing problem, Ford and Fulkerson (1956) for the maximum flow problem .
- Bernard Roy (1958) developed the MPM method for the scheduling problem.

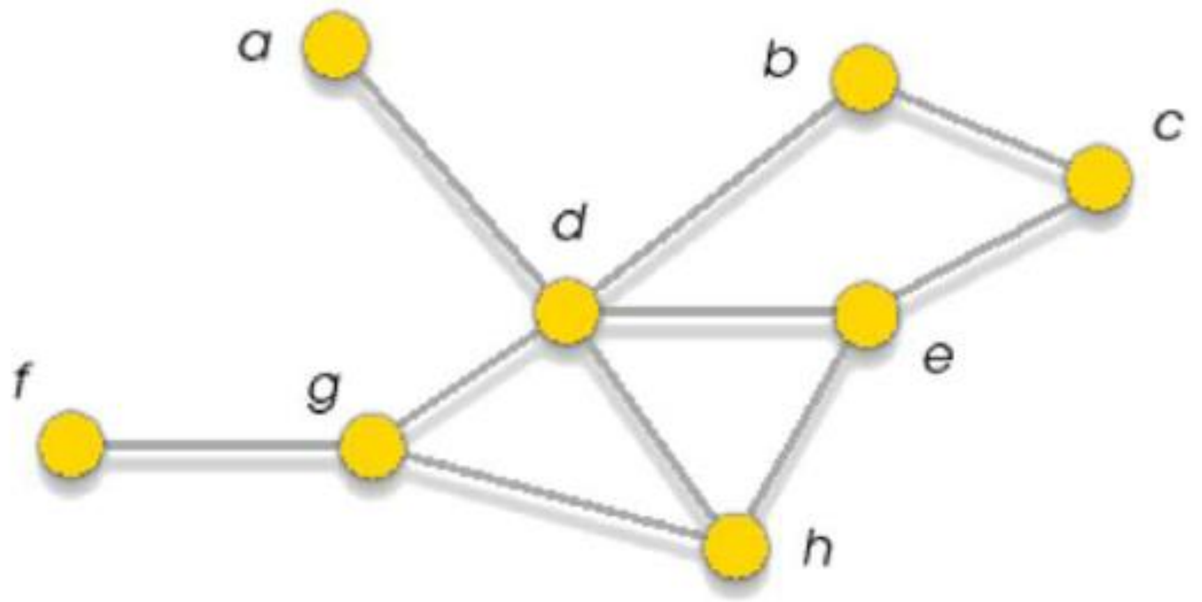
Graph Theory

- Graph theory is a powerful set of tools for modeling and solving real-world problems.
- **Definition** :
  - A Graph is a collection of vertices (nodes) and edges (links) connecting pairs of vertices.
  - A graph is used to describe a set of objects and their relationships, that is to say the links (edges) between the objects (nodes /vertices).
  - A graph (G), denoted as (G = (X, U)), is defined by:
    - **A finite set of vertices (X)**: The fundamental units or points in the graph.
    - **A finite set of edges/arcs (U)**: Each edge/arc connects a pair of vertices in (X).

Give me some examples….

Undirected graph

- An undirected graph G is a pair (V, E).
- V is a (finite) set of objects. The elements of V are called the vertices of the graph.
- E is a subset of VxV . The elements of E are called the edges of the graph.
- An edge "e" of the graph is a pair e=(x , y) = (y, x) of vertices.
- Vertices x and y are the endpoints of the edge.

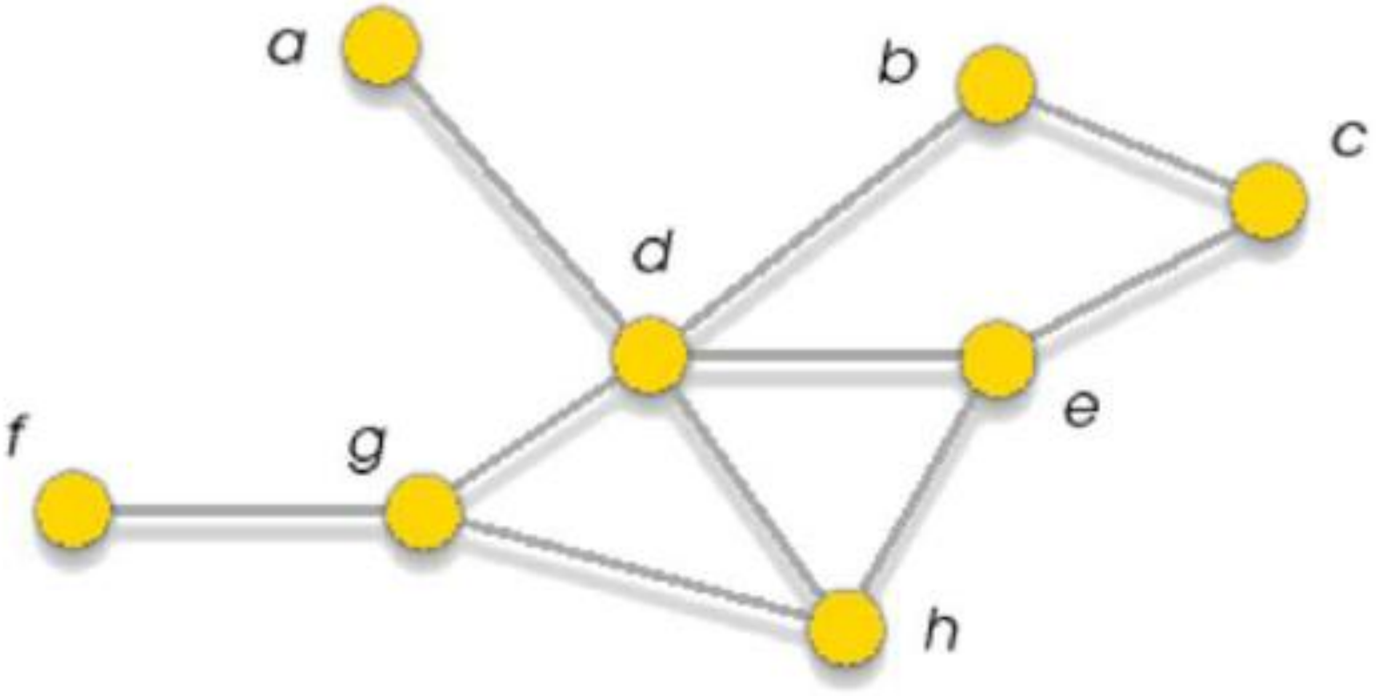


$$E = \{......\}$$
$$V = \{......\}$$

I.2. Degree of a vertex/graph

- Two vertices ***x*** and ***y*** are **adjacent** if there exists the edge ***(x,y)*** in ***E*** . The vertices ***x*** and ***y*** are then said to be **neighbors**
- An edge is **incident** to a vertex ***x*** if ***x*** is one of its endpoints.
- The **degree** of a vertex ***x*** of ***G*** is the number of edges incident to ***x*** . It is denoted **d *(x)*** .
- For a simple graph the degree of ***x*** also corresponds to the number of vertices adjacent to ***x*** .
- For a simple graph of order ***n,*** the degree of a vertex is an integer between ***0*** and ***n-1***
- A vertex of degree ***0*** is said to be ***isolated*** : it is not connected to any other vertex.

Example



Vertex	Degree
a	1
b	2
c	
d	5
e	
f	
g	
h	

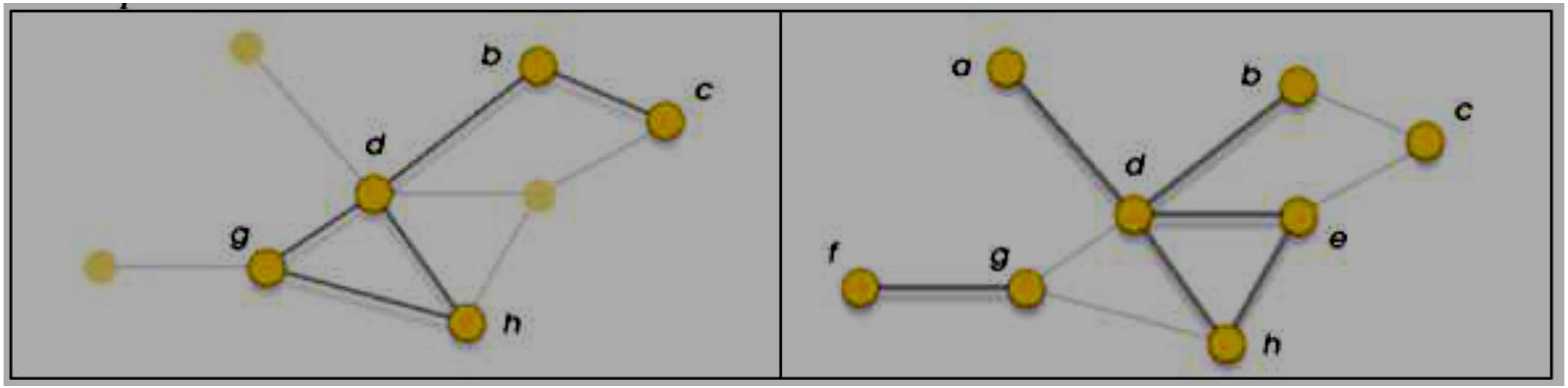
Degree of a vertex/graph

- **Property 1**: The sum of the degrees of the vertices of a graph is equal to 2 times its number of edges. $\sum_{x \in V} d(x) = 2 * ||E||$

- **Property 2**: The number of odd-degree vertices in a graph is even.
  - The degree of a graph is the maximum degree of all its vertices.
  - A graph whose vertices all have the same degree is said to be regular .
  - If the common degree is k , then the graph is said to be k-regular.

Subgraph and partial graph

- To characterize the structure of a graph in a less local way, it is possible to search for remarkable parts of the graph, by restricting either :
  - The set of vertices (**subgraph**), or
  - The set of edges (**partial graph**).



Subgraph

- **Definition** : A subgraph is formed from a subset of a graph's vertices and edges.
- Formally : For a graph ***G = (V, E)***
  A **subgraph** of ***G*** is a graph ***H=(W, E(W))*** such that ***W*** is a subset of ***V*** , and ***E(W)*** are the edges induced by ***E*** on ***W*** , that is to say the edges of ***E*** whose 2 extremities are vertices of ***W*** . ,

  $E(W)= \{(X,Y) \in E /, X,Y \in W\}$
- A *subgraph* of ***G*** consists of considering only a part of the vertices of ***V*** and the links induced by ***E.***
- For example, if ***G*** represents the daily air connections between the main cities of the world, a possible subgraph is to restrict itself to the daily connections between the main European cities.

Partial graph

- A *partial graph* of ***G*** consists of considering only a part of the edges of ***E.***
- Retains all the vertices of the original graph while having a reduced number of edges.
- The vertex set remains identical to that of the original graph,
- The edge set is a subset, meaning some connections between the vertices may be missing.
- A **partial graph** of ***G =(V, E)*** is a graph ***I=(V,F)*** such that ***F*** is a subset of ***E.***
- Example : a possible partial graph is to consider only the daily connections of less than 3 hours between the main cities of the world.

Complete graph

- **Definition:** A complete graph is a type of graph in which every pair of distinct vertices is connected by an edge. In other words, there is an edge between every two vertices,
- Ensuring that each vertex has a direct connection to every other vertex in the graph.
- In a complete graph of order ***n*** is denoted $\boldsymbol{K_n}$:
  - Each vertex is of degree ***n-1*** .
  - The number of edges in a complete graph is: **n*(n-1) / 2**

Clique

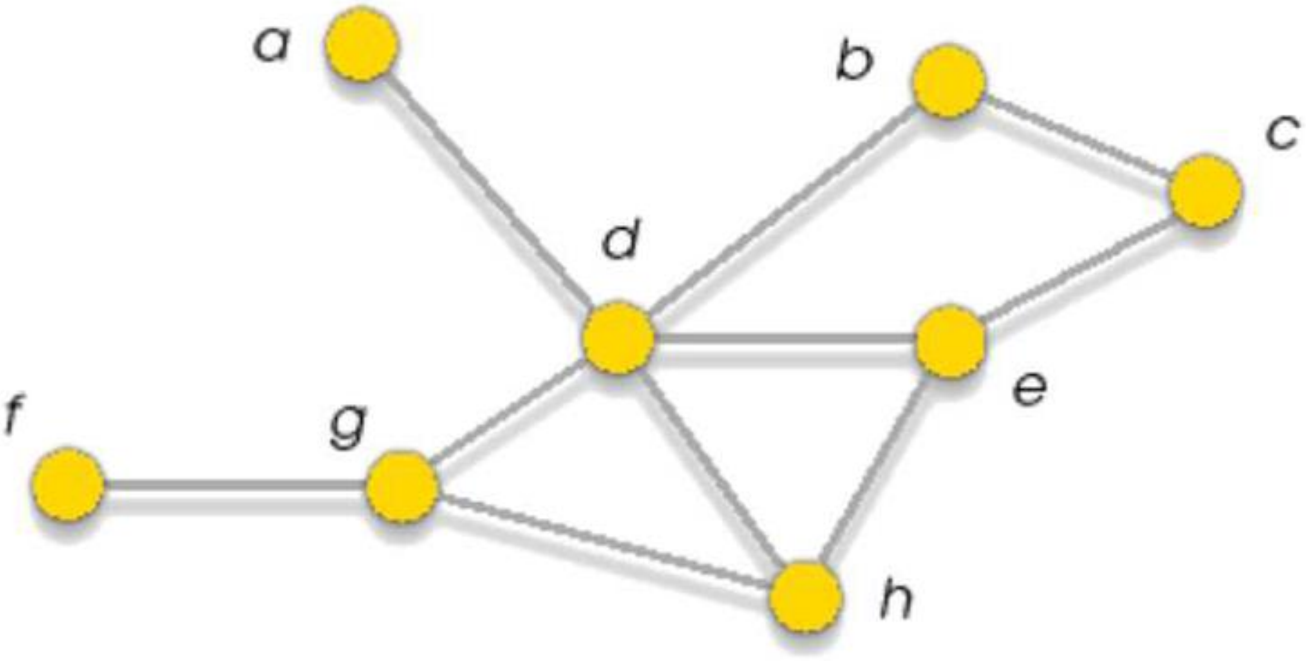
- A clique in a graph is a subset of vertices where each pair of vertices is directly connected by an edge, forming a complete subgraph.
- It is a maximal complete subgraph:
  - Meaning that no more vertices can be added to the clique without losing the property of complete connectivity.
  - In other words, adding more vertices would break the condition that every two vertices in the clique are adjacent.
- The size of a clique is the number of vertices it contains.
- Example: In a social network, a clique could represent a group of people who all know each other directly.

Stable (Independent Set)

- A **stable** or independent set is a subset of vertices such that no two vertices in the subset are adjacent (i.e., there are no edges between any pair of vertices in the set). <u>This is the opposite of a clique</u>.
- A stable is a <u>subgraph</u> without edges.
- **Applications**: Independent sets are relevant in scheduling problems, where tasks (represented by vertices) that conflict (connected by edges) cannot be scheduled simultaneously.
- **Example** : In a map coloring problem, vertices of the same color form an independent set, as no two adjacent regions can have the same color.

Example

- *2 cliques of order 3 defined by the sets of vertices* ***{ d, g, h }*** and ***{ d, e, h }***
- *4 stable sets of order 4 defined by the sets of vertices* ***{ a, b, e, g }*** *and* ***{ a, b, h, f } { a, c, h, f }*** *and* ***{ a, b, e, f }***



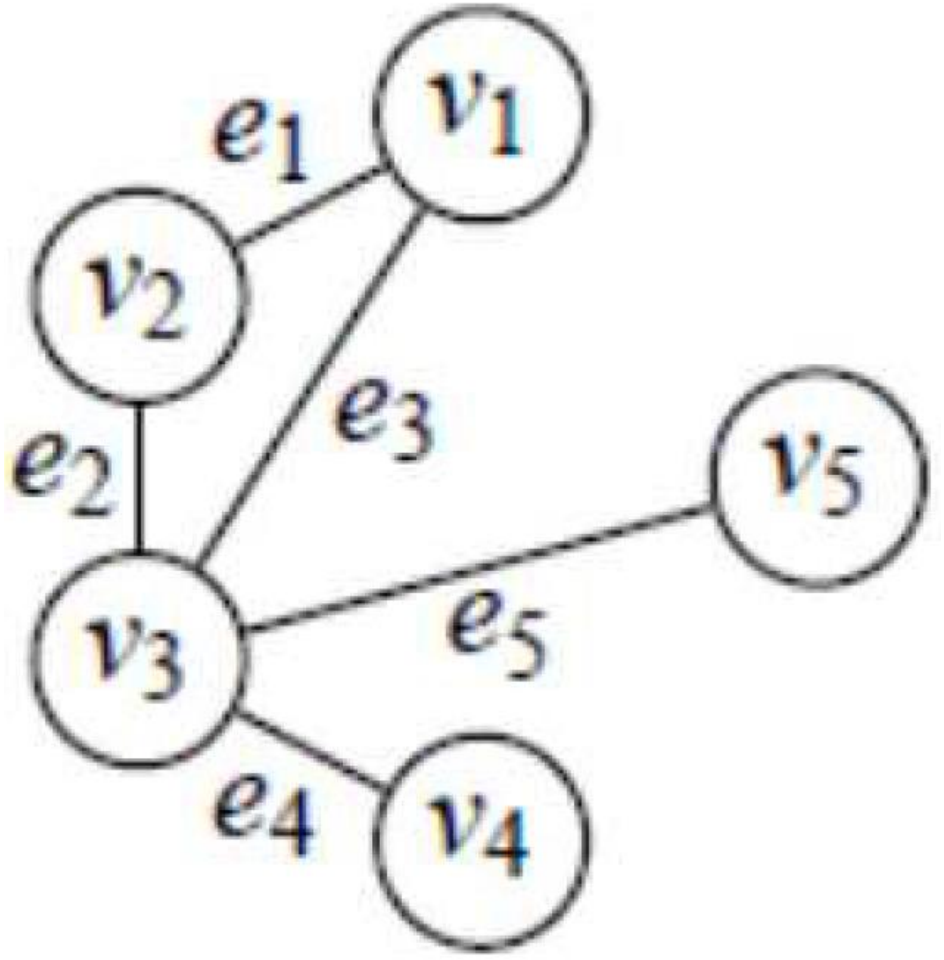
Chain 1/4

- **Definition:** A **chain** in G is a sequence having as its elements alternately **vertices** and **edges**, starting and ending with a **vertex**, and such that each edge is framed by its endpoints.
- We will say that the chain **connects** the first vertex of the sequence to the last vertex.
- The chain has the **length** of the number of edges in the chain.

Chain (Example) 2/4

- The following graph contains among others the chains (*v* 1, *e* 1, *v* 2, *e* 2, *v* 3, *e* 5, *v* 5) and (*v* 4, *e* 4, *v* 3, *e* 2, *v* 2, *e* 1, *v* 1).

- A string is not changed by reversing the order of the elements in the corresponding sequence. Thus, the strings (*v* 1, *e* 3, *v* 3, *e* 4, *v* 4) and (*v* 4, *e* 4, *v* 3, *e* 3, *v* 1) are identical.



Chain (Definition) 3/4

- **The distance between two vertices** is called the length of the smallest chain connecting them.
- **The diameter of a graph** is called the longest distance between two vertices.
- A chain is **elementary** if each vertex appears in it at most once.

  Property: In a graph ***G*** of order ***n***

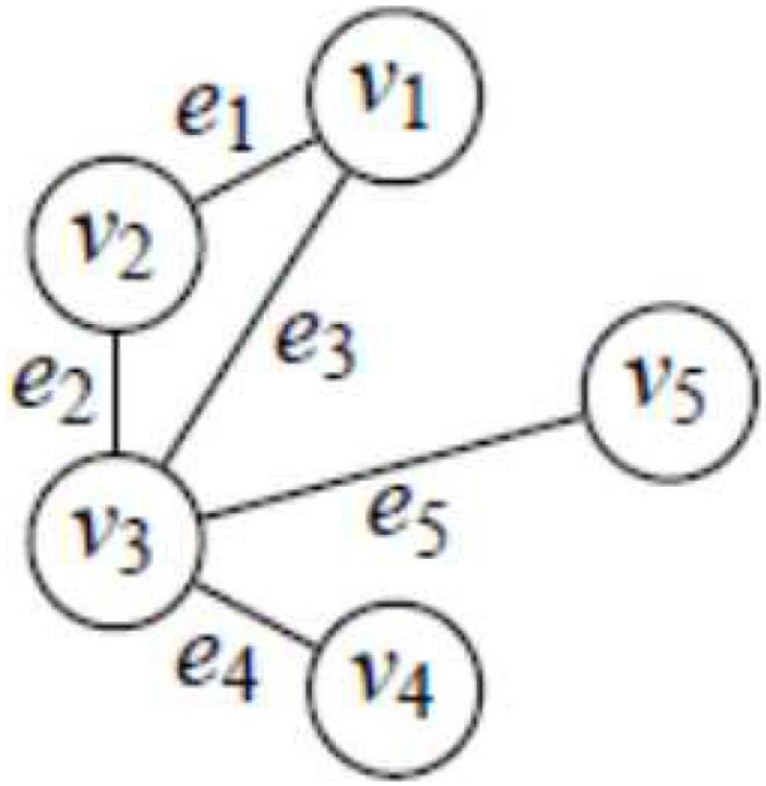
  - Any elementary chain is of length at most ***n*- 1**
  - The number of elementary chains in the graph is finite.

Chain (Definition) 4/4

- A chain is **simple** if each edge appears at most once.
  In the previous graph, (v1, e1, v2, e2, v3) is a simple, elementary chain.
- A chain whose starting and ending vertices are the same is called a **closed chain**.
- *In the previous graph, (v4, e4, v3, e5, v5, e5, v3, e4, v4) is a closed chain.*

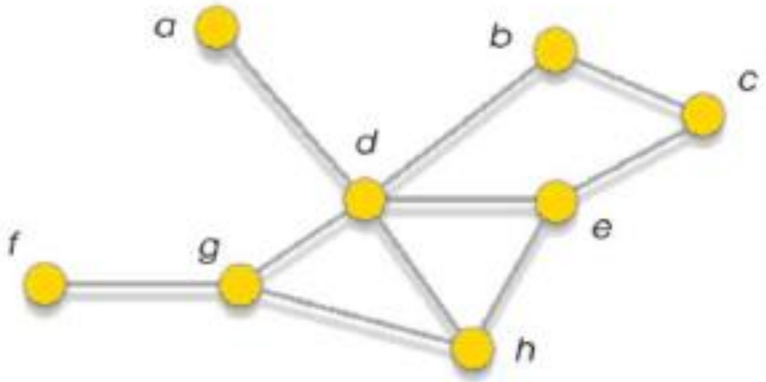
Cycle

- A simple closed chain is called **a cycle** .
  In the following graph, the chain (v1, e1, v2, e2, v3, e3, v1) is a cycle.



Connectivity of a graph

- **Definition:** A graph is **connected** if and only if there exists a chain between every pair of vertices.
- If the graph ***G*** is not connected, then it appears as a set of connected graphs.
- Each of these graphs is a particular subgraph of ***G*** , called ***a connected component*** .
- A connected Graph has one ***connected component.***

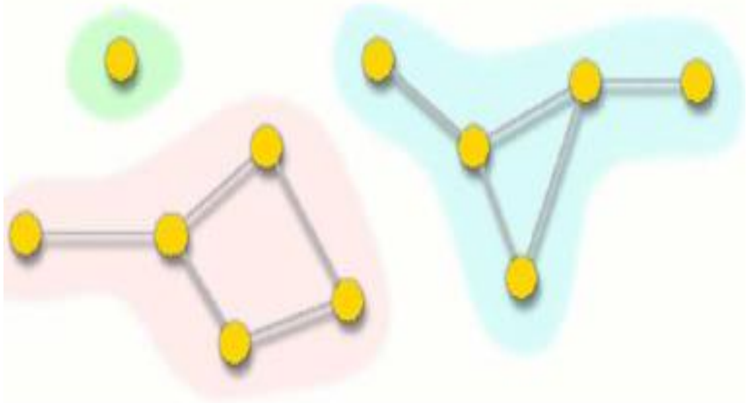


Connectivity of a graph

Definition: A **connected component** of a graph ***G*** is a maximally connected subgraph ***G'=(V',E')*** : it is not possible to add other vertices to ***V'*** while preserving the connectivity of the subgraph.

- A graph with only one connected component is simply a connected graph.
- An isolated vertex (of degree 0) always constitutes a connected component on its own.
- The relation on the vertices "there exists a chain between ..." is an equivalence relation (reflexive, symmetric and transitive). The connected components of a graph correspond to the equivalence classes of this relation.
  - All vertices within the same connected component are related to each other by the existence of paths (chains), forming a single equivalence class.

Property: A connected graph ***G*** of order ***n*** has at least ***n- 1*** edges.



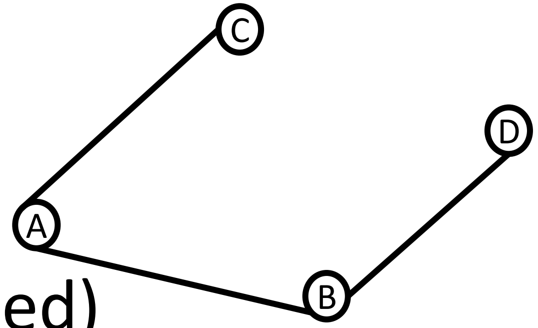
Acyclic graph

A graph in which there are no cycles is said to *be acyclic* .

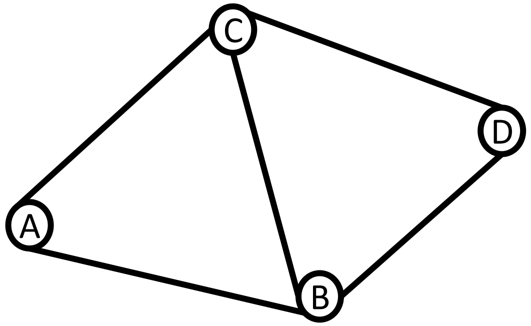
- *Property 1:* If in a graph ***G*** every vertex is of degree greater than or equal to 2, then ***G*** has at least one cycle.
  - This simple property implies that a graph without a cycle has at least one vertex of degree 0 or 1.
- *Property 2:* An acyclic graph ***G*** with ***n*** vertices has at most ***n-* 1** edges.

Acyclic graph

- **Definition:** For a graph *G* having *m* edges, *n* vertices and *p* connected components, we define: n(*G*) = *m* − *n* + *p.*
- n(*G*) is called the **cyclomatic number** .
- We have an (*G*) > 0 for any graph *G* .
- n(*G*) = 0 if and only if *G* is cycle-free.



- *m: 4*
- *n* : 3
- *p* : 1 (the graph is connected)
- n(*G*) = 3 − 4 + 1 = 0.
- This graph is cycle-free (**Acyclic graph**)



- *m: 4*
- *n* : 5
- *p* :1 (the graph is connected)
- n(*G*) = 4 − 5 + 1= 2 >0.