Graph theory

Graph coloring

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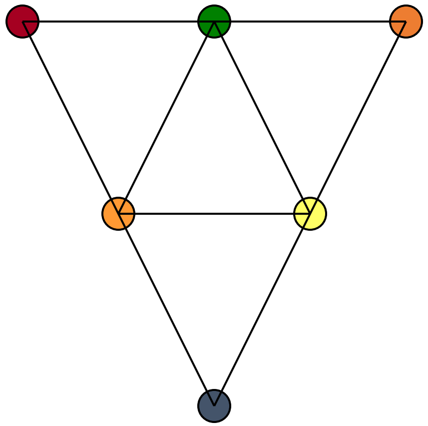
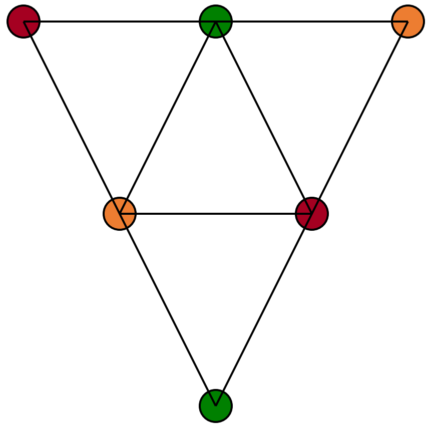
Graph Coloring

- Graph coloring can be described as the process of assigning labels (or colors) to the vertices of a graph.
- A proper graph coloring is an assignment of colors to the vertices of a graph such that no two adjacent vertices have the same color.
- We say that G is k-colorable.
- Formally: For a graph G = (V, E), where V is the set of vertices and E is the set of edges,
    - a proper coloring is a function $c: V \rightarrow S$,
    - where S is a set of colors,
    - such that $c(u) \neq c(v)$ for all edges $(u, v) \in E$.

Chromatic number

- The chromatic number of a graph G is the smallest number of colors needed to achieve a proper coloring of the graph.





3-colourable

Objective: use minimum number of colors .

Chromatic number

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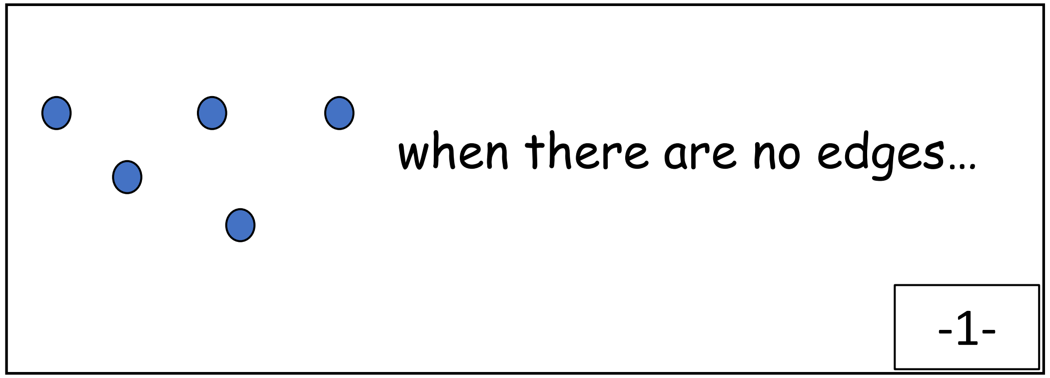

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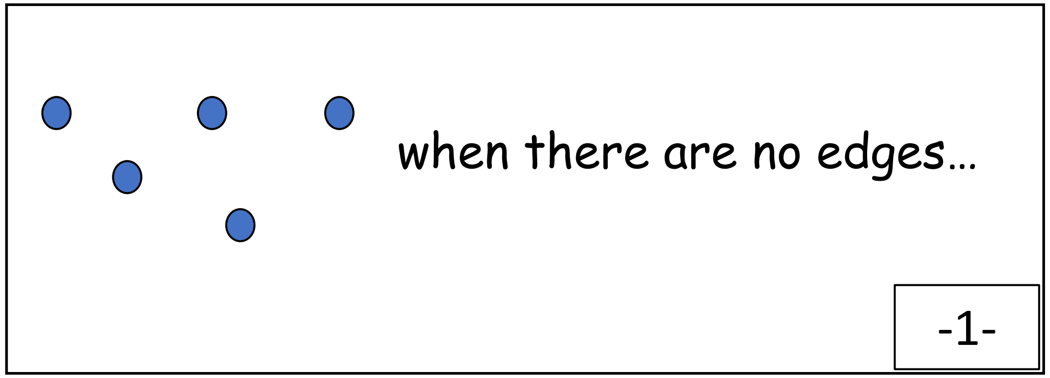

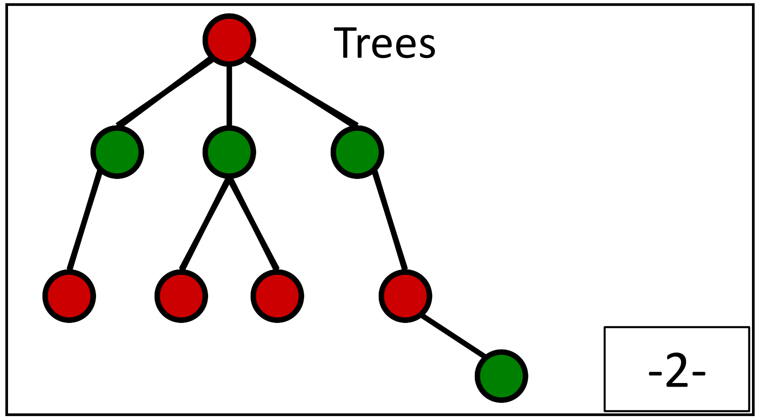

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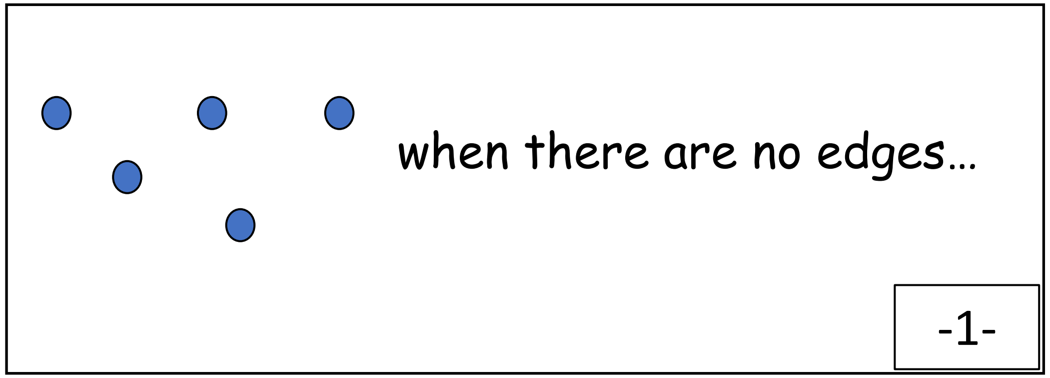
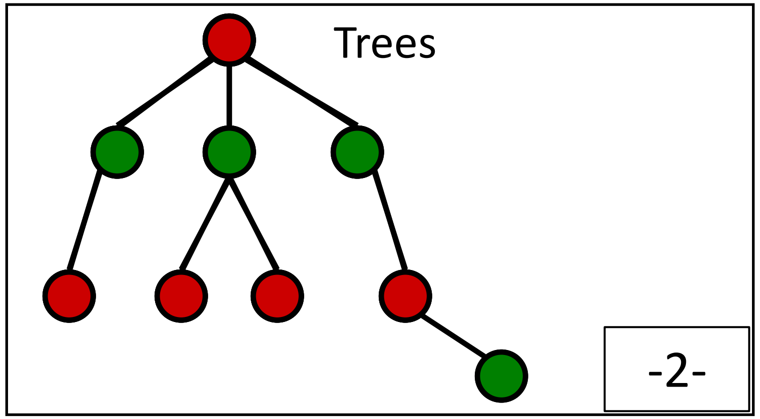
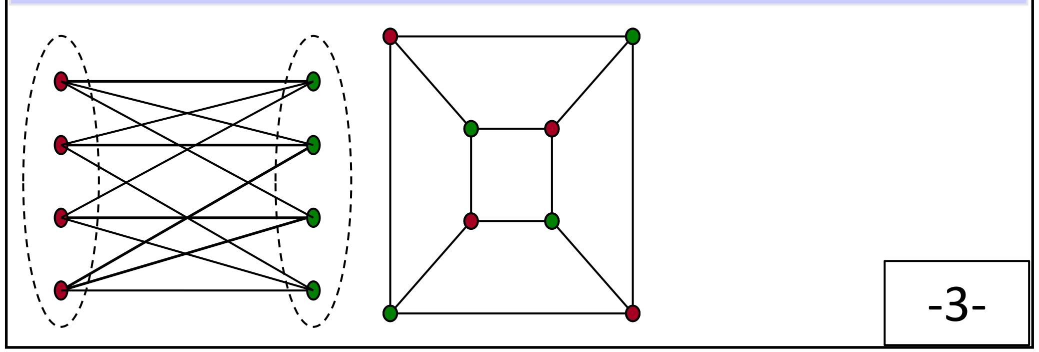




A graph is 2-colorable if and only if it is bipartite.



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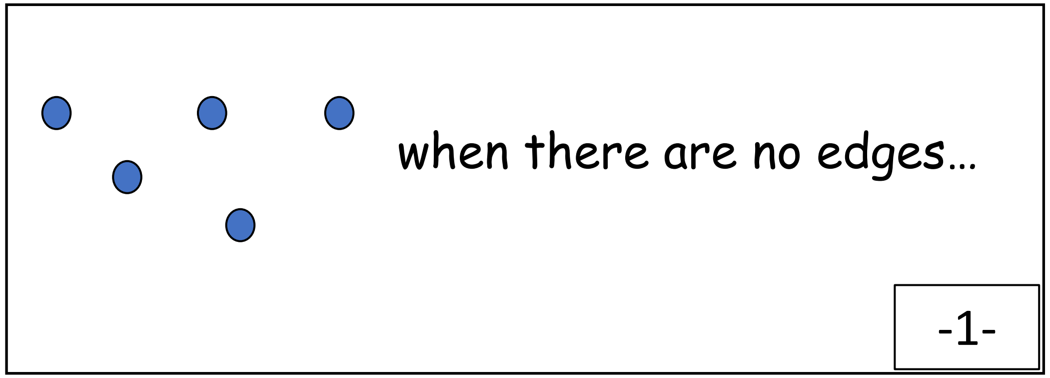
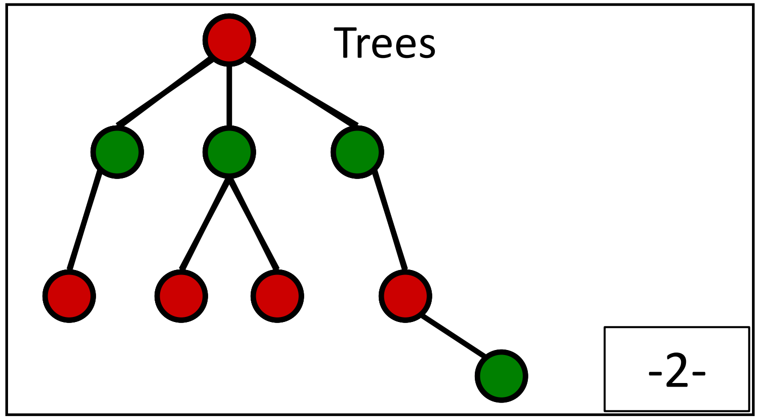
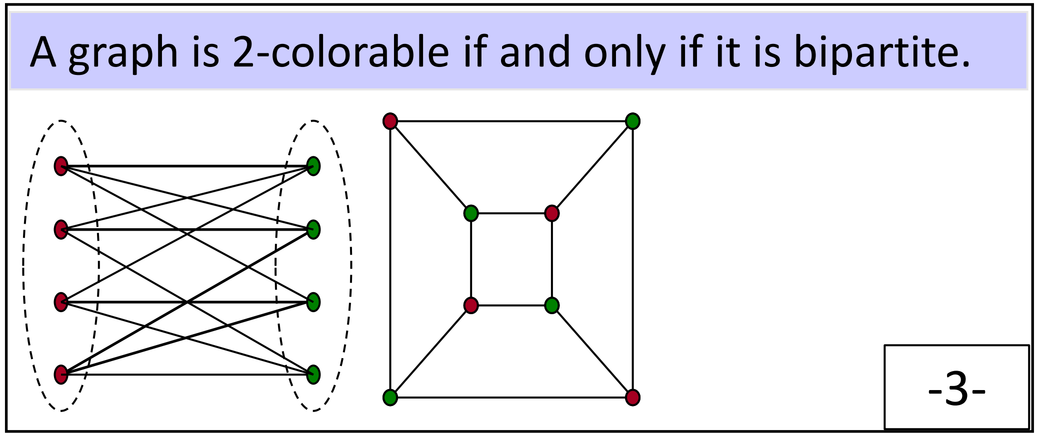
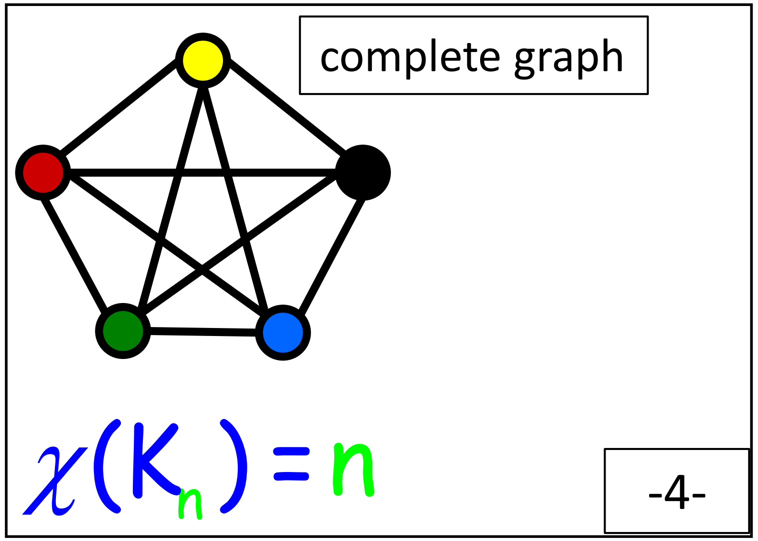








Greedy Coloring

- Colors each vertex in a sequential order, assigning the smallest possible color that hasn't been used by its adjacent vertices.
- **Positive aspects** :
  - Simple and easy to implement.
  - Runs in linear time with respect to the number of vertices and edges.
- **Negative aspects** :
  - Depends heavily on the vertex ordering. It can use more colors than necessary if the vertices are not ordered optimally.
  - May produce suboptimal colorings (i.e., use more than the chromatic number).

Welsh and Powell algorithm 1/2

- A variation of the greedy algorithm that sorts vertices by degree (number of neighbors) before coloring.
- **Algorithm:**
  1. Sort vertices in descending order of degree.
  2. Assign the first color to the first vertex in the ordered list **and** to every vertex not adjacent to it.
  3. Repeat step 2 with a new color for the uncolored vertices until all vertices are colored.

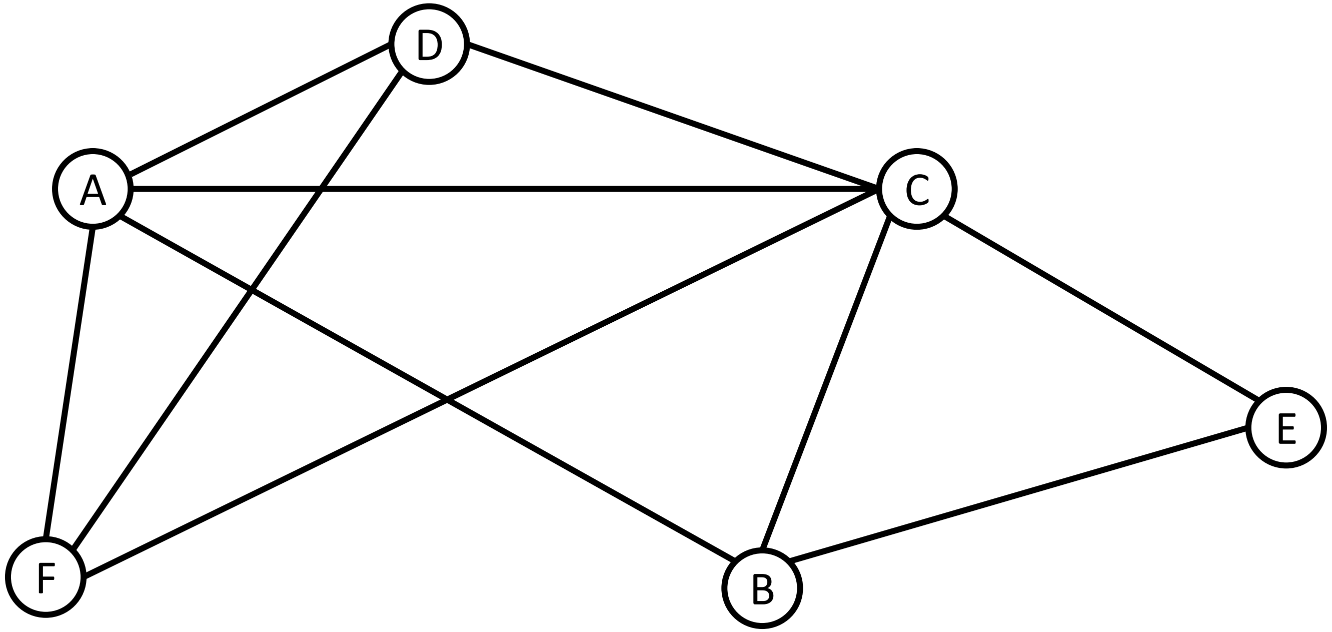
Welsh and Powell algorithm 2/2

- **Positive aspects** :
  - Generally, produces better colorings than the basic greedy algorithm since higher-degree vertices are colored first.
  - Simple to implement and has better performance for dense graphs.
- **Negative aspects** :
  - Still not guaranteed to find the chromatic number.*
  - It is affected by the tie-breaking method when vertices have the same degree.

Example

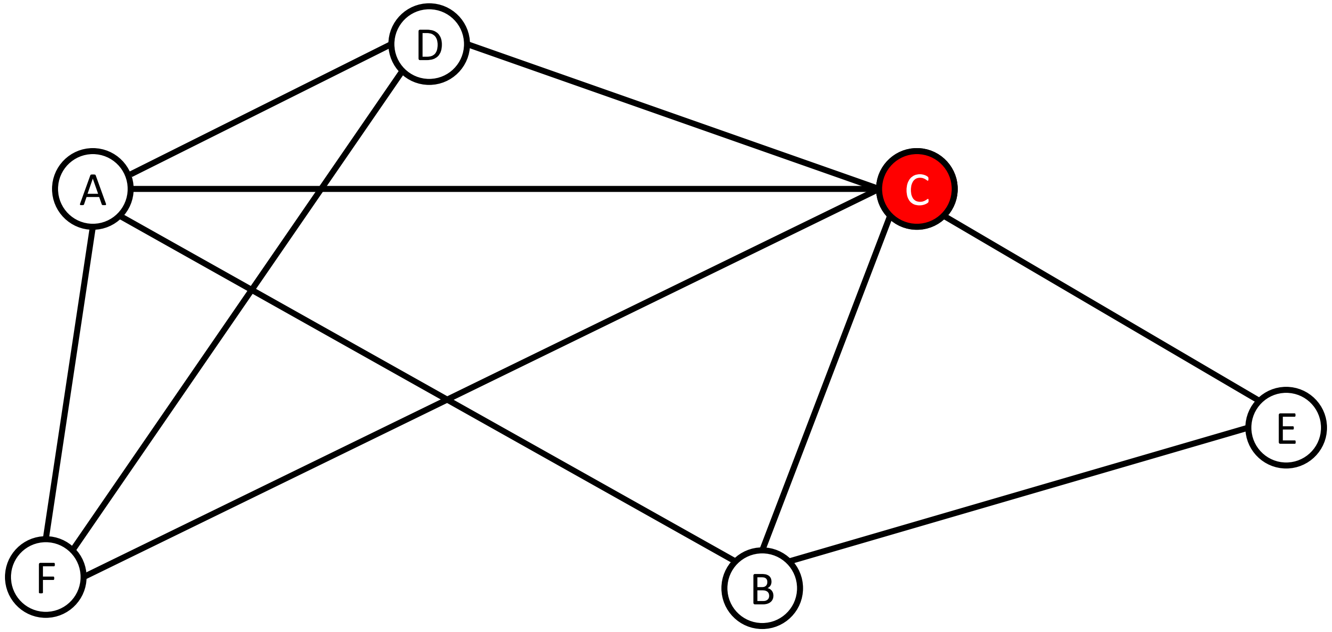
- Five students have to take written exams.
  - Student 1: A, D, C
  - Student 2: E, B, C
  - Student 3: C, F, A
  - Student 4: A, B
  - Student 5: D, F
- If each writing lasts 1/2 day, what is the minimum number of days that should be planned?

Welsh and Powell



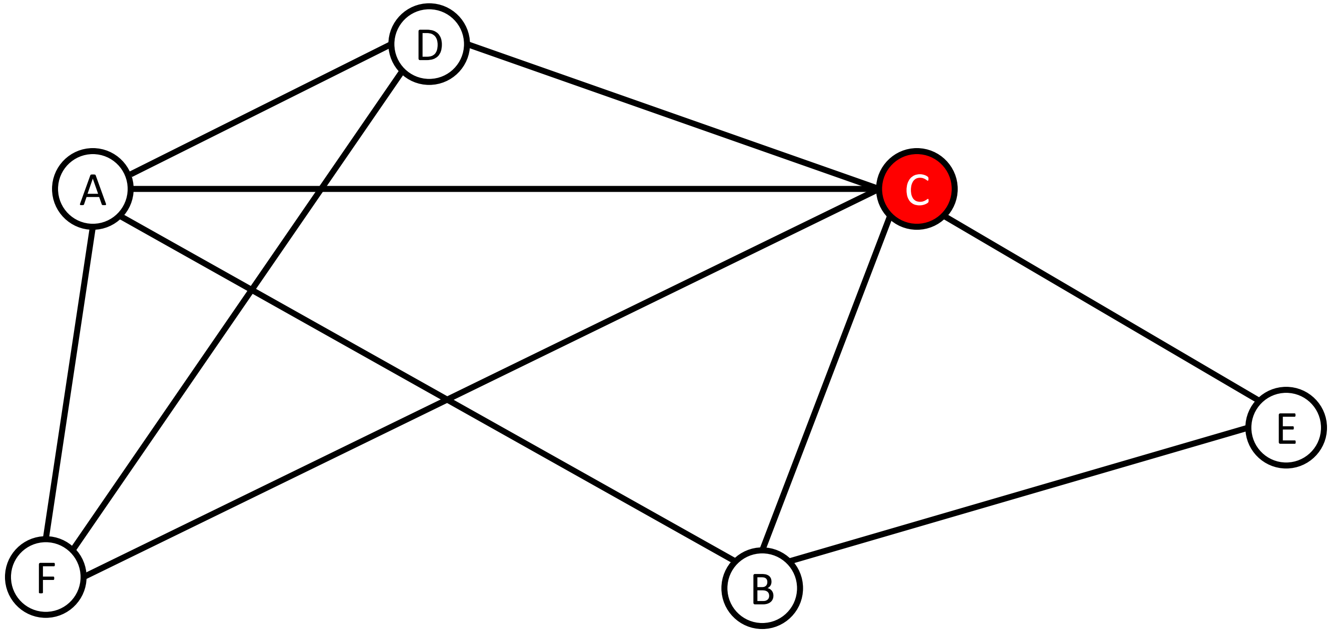
Summit	C	A	D	F	B	E
Degree	5	4	3	3	3	2
Color						

Welsh and Powell



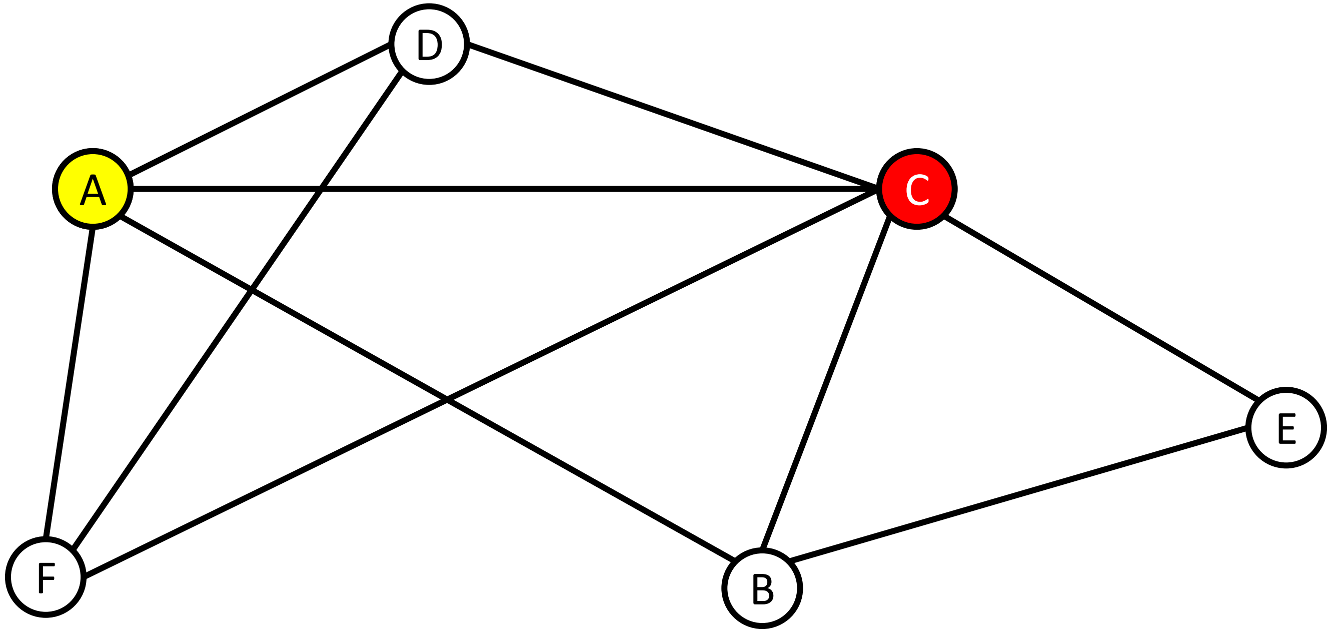
Summit	C	A	D	F	B	E
Degree	5	4	3	3	3	2
Color	Red					

Welsh and Powell



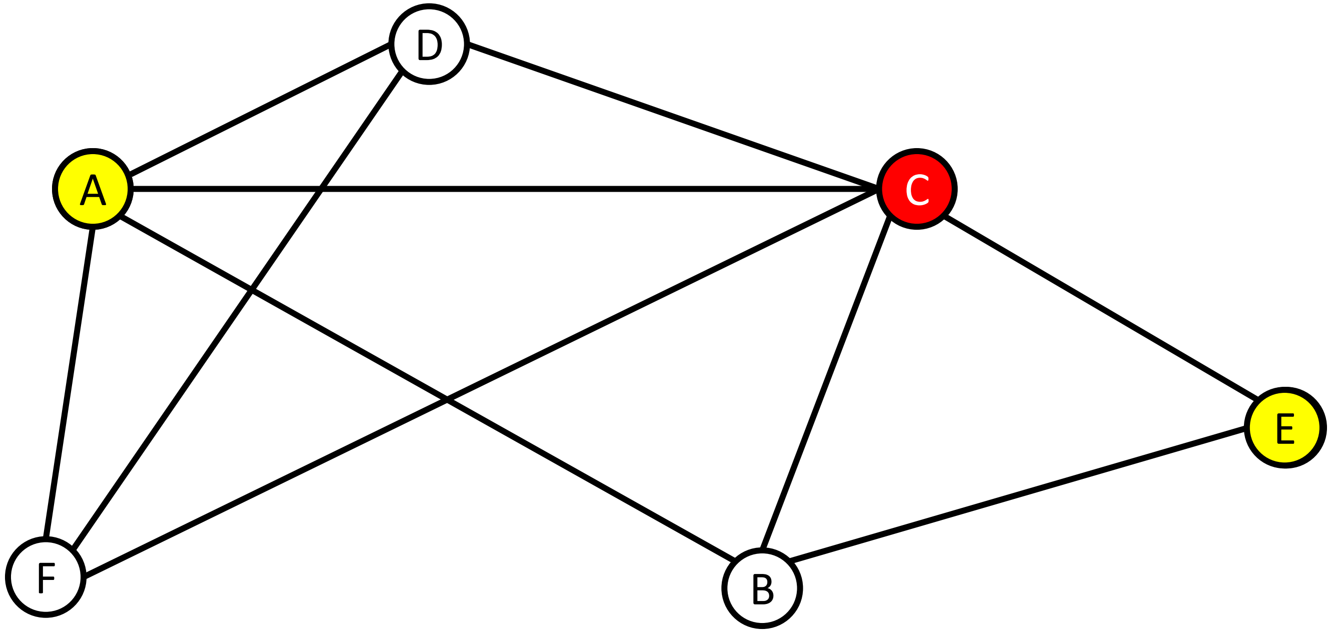
Summit	C	A	D	F	B	E
Degree	5	4	3	3	3	2
Color	Red					

Welsh and Powell



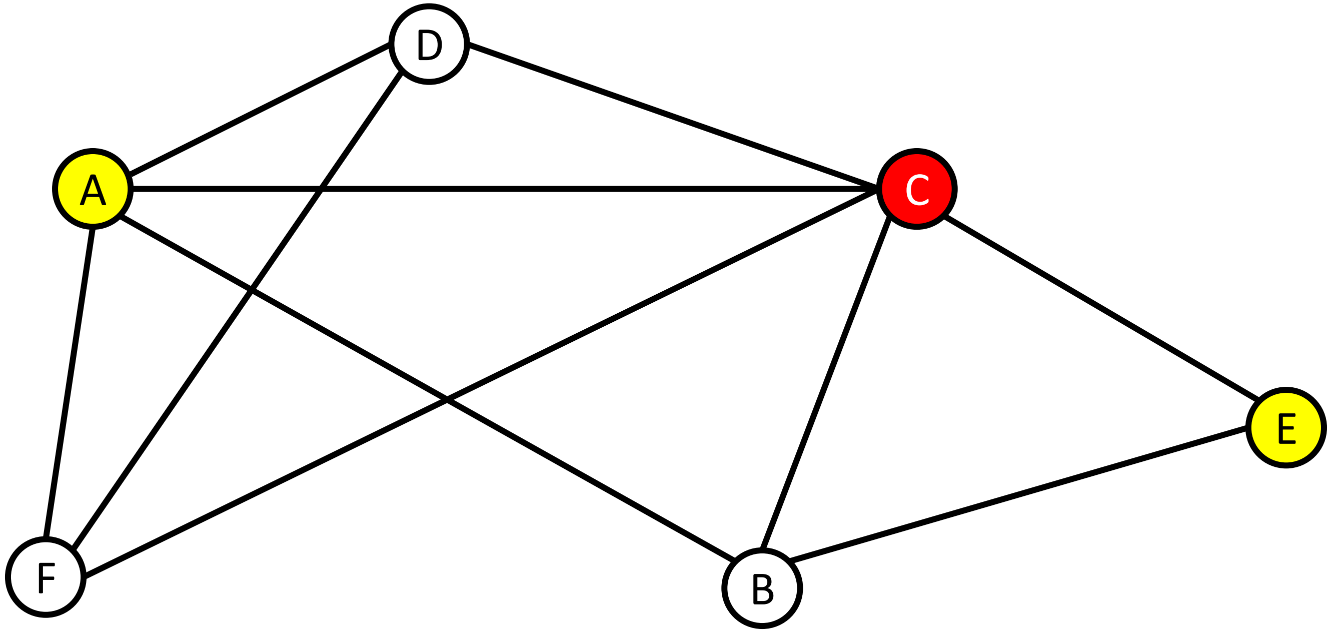
Summit	C	A	D	F	B	E
Degree	5	4	3	3	3	2
Color	Red	YEL				

Welsh and Powell



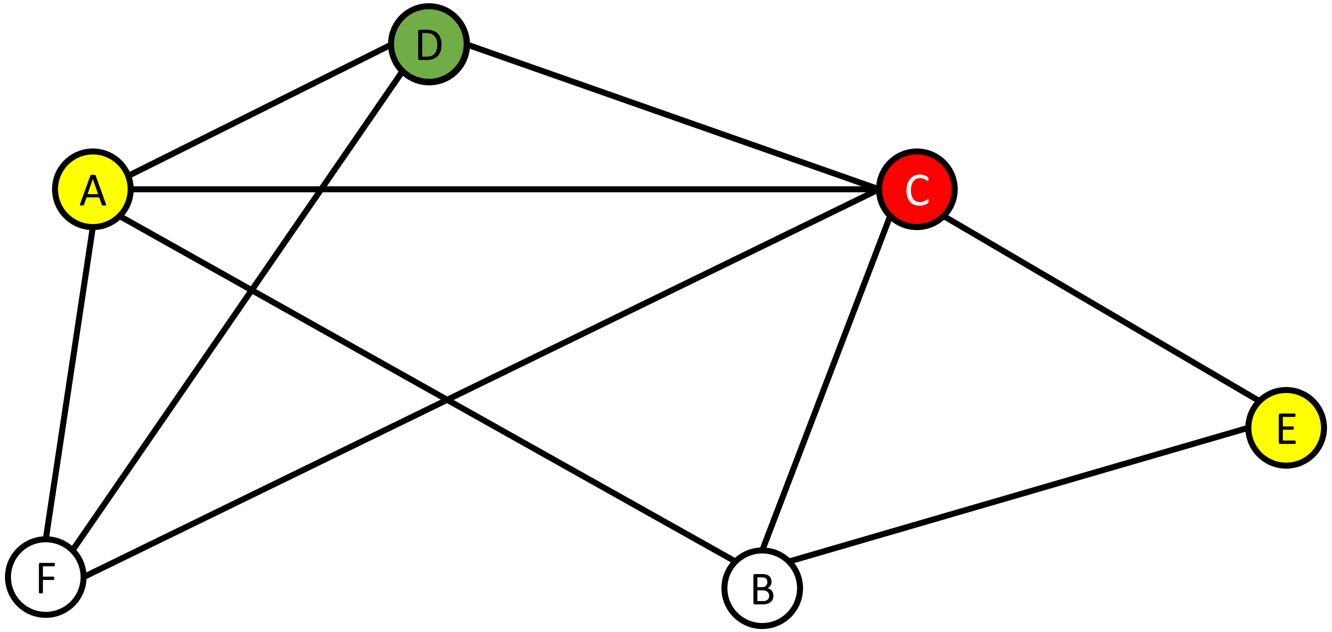
Summit	C	A	D	F	B	E
Degree	5	4	3	3	3	2
Color	Red	YEL				YEL

Welsh and Powell



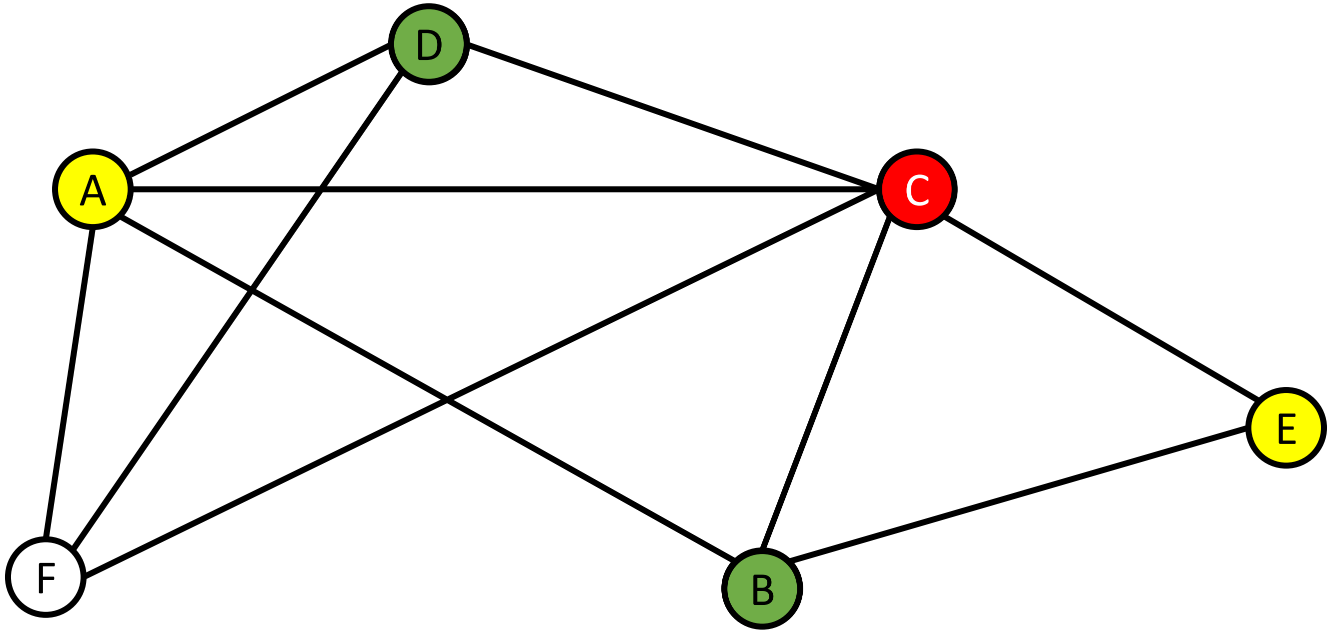
Summit	C	A	D	F	B	E
Degree	5	4	3	3	3	2
Color	Red	YEL				YEL

Welsh and Powell



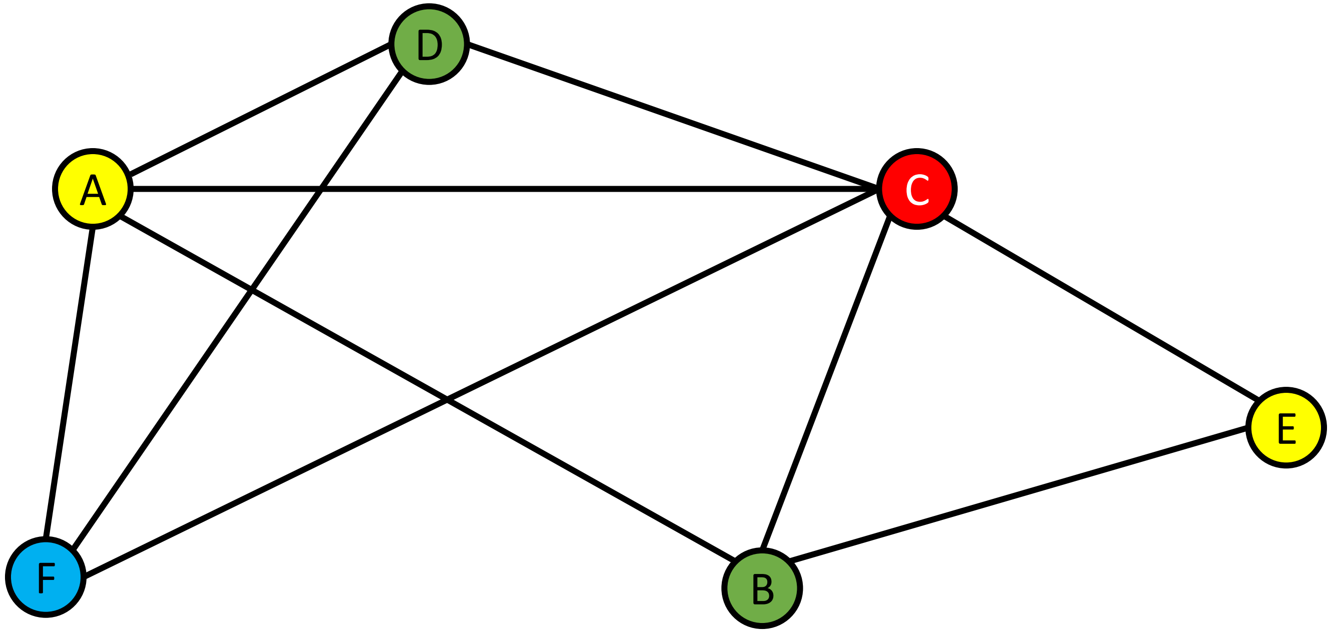
Summit	C	A	D	F	B	E
Degree	5	4	3	3	3	2
Color	Red	YEL	GR			YEL

Welsh and Powell



Summit	C	A	D	F	B	E
Degree	5	4	3	3	3	2
Color	Red	YEL	GR		GR	YEL

Welsh and Powell



Summit	C	A	D	F	B	E
Degree	5	4	3	3	3	2
Color	**Red**	**YEL**	**GR**	**BLUE**	**GR**	**YEL**

DSatur Algorithm 1/2

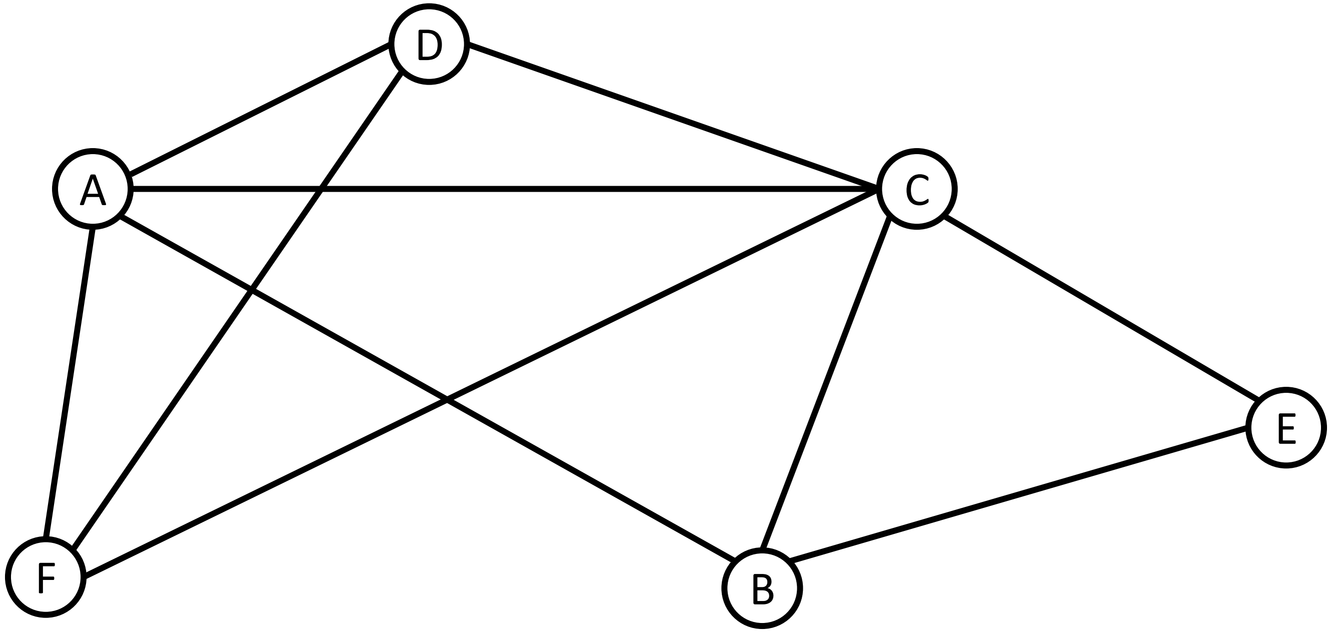
- Colors vertices based on saturation degree, prioritizing those with the highest number of differently colored neighbors.
- **Algorithm:**

1. Order vertices by degree.
2. Color a vertex of maximal degree with the first color.
3. Choose a vertex with maximal saturation degree. If there's a tie, choose the vertex with highest degree in the uncolored subgraph.
4. Color the chosen vertex with the lowest possible color number.
5. Repeat steps 3-4 until all vertices are colored.

DSatur Algorithm 2/2

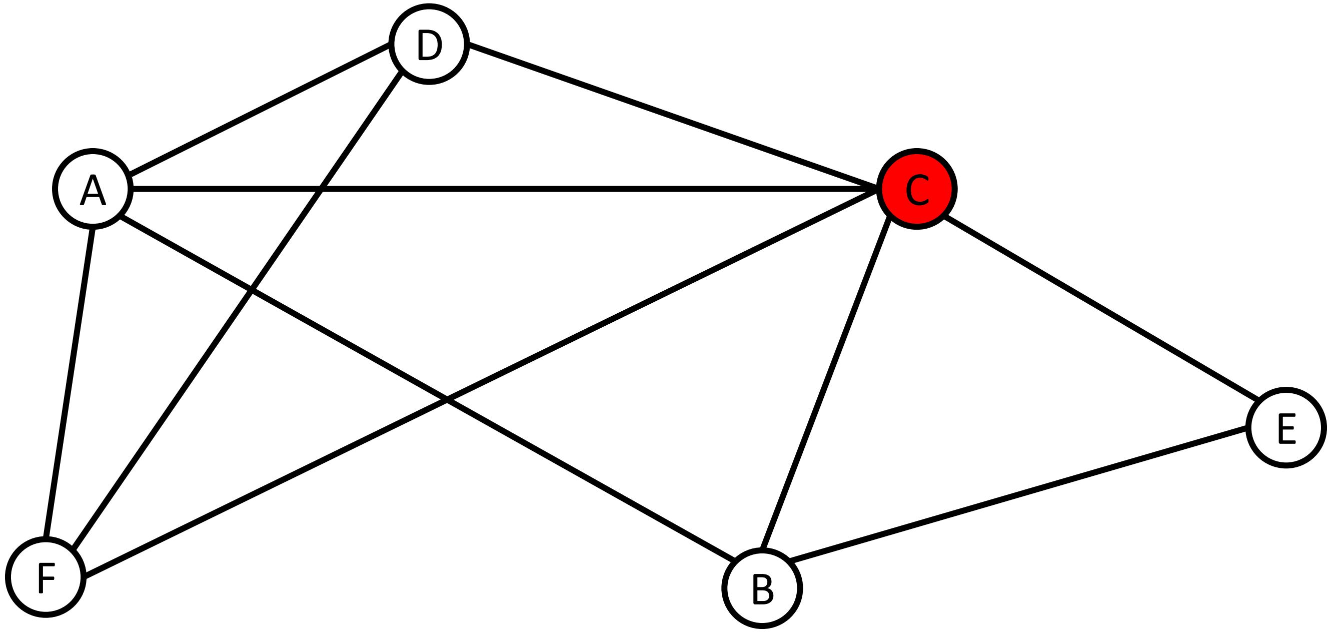
- **Positive aspects** :
  - Generally produces better colorings than Greedy or Welsh-Powell
  - Tends to produce colorings closer to the chromatic number, especially for dense graphs.
  - Adapts to the graph structure during the coloring process
- **Negative aspects** :
  - Slightly more complex to implement due to the need to track saturation degrees and dynamically update them after each coloring step.
  - Heuristic algorithms like DSatur are designed to find good solutions quickly, but they may not always find the best possible solution.

DSatur Algorithm



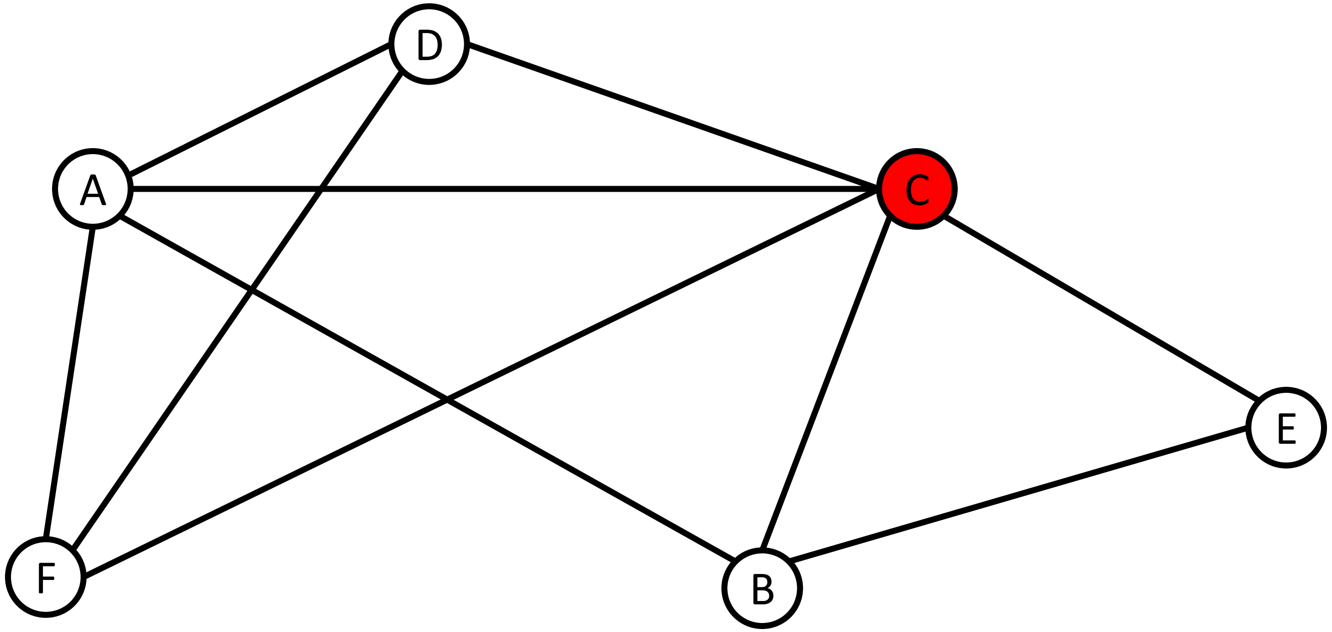
Summit	C	A	D	F	B	E
Degree	5	4	3	3	3	2
Color						
saturation degree	0	0	0	0	0	0

DSatur Algorithm

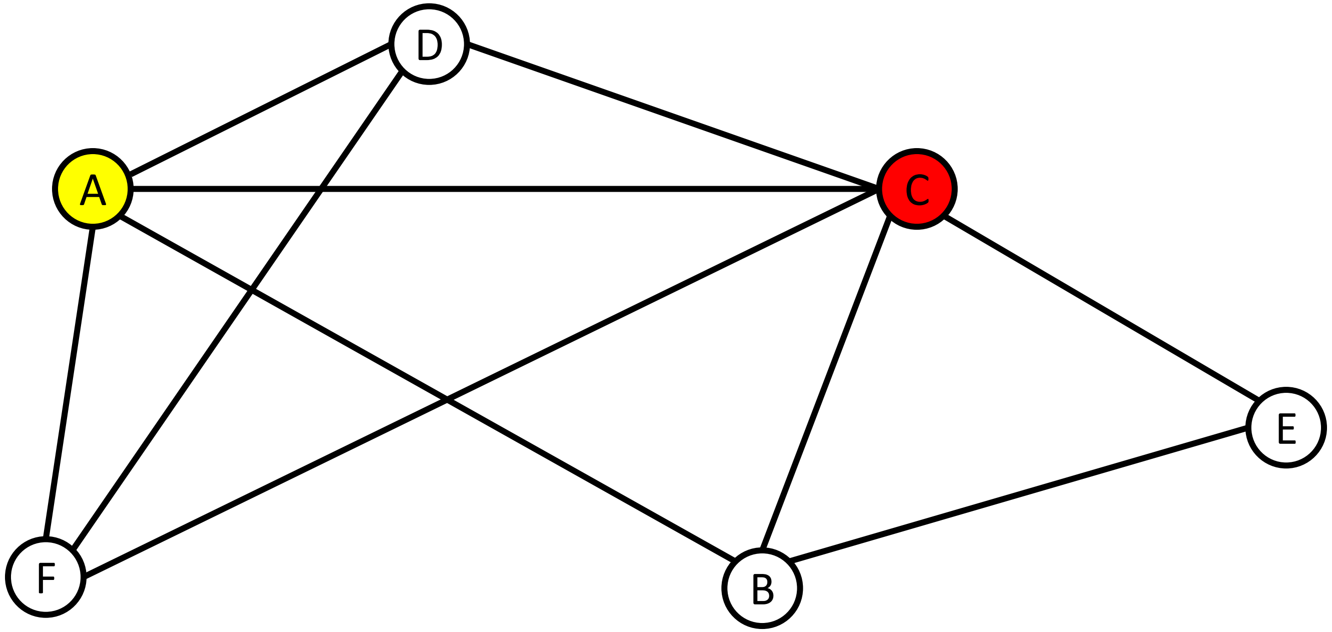


Summit	C	A	D	F	B	E
Degree	5	4	3	3	3	2
Color	Red					
saturation degree	0	0	0	0	0	0

DSatur Algorithm

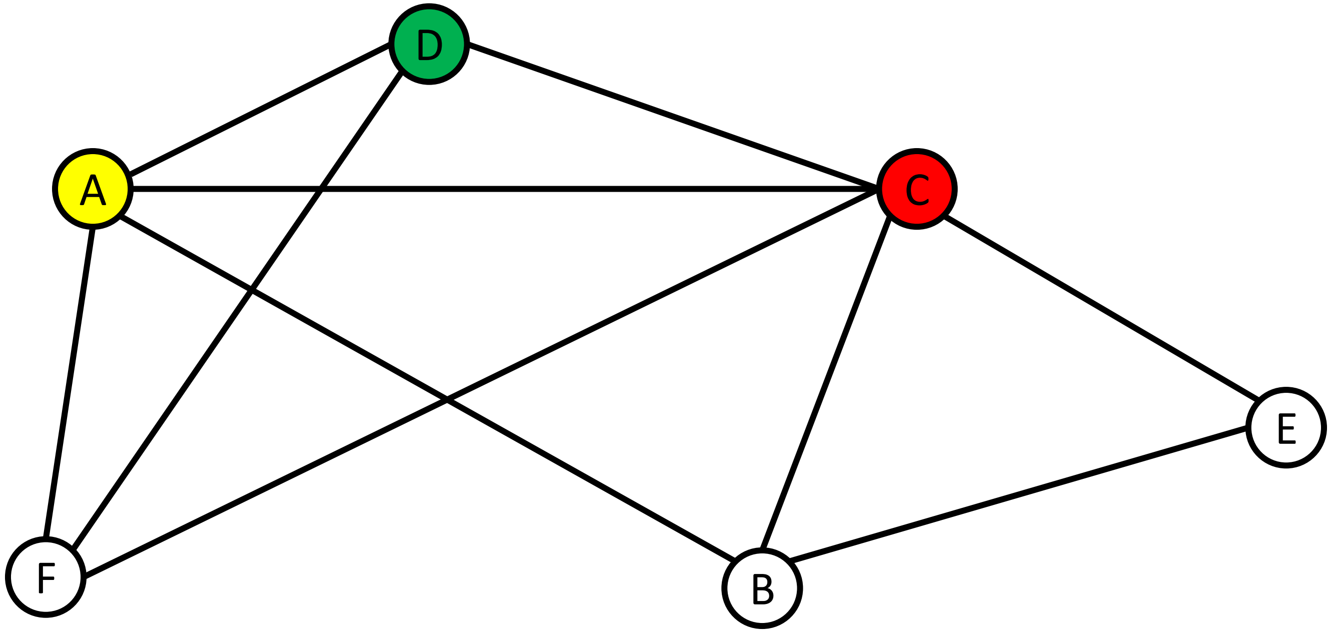


Summit	C	A	D	F	B	E
Degree	5	4	3	3	3	2
Color	Red					
saturation degree	0	1	1	1	1	1



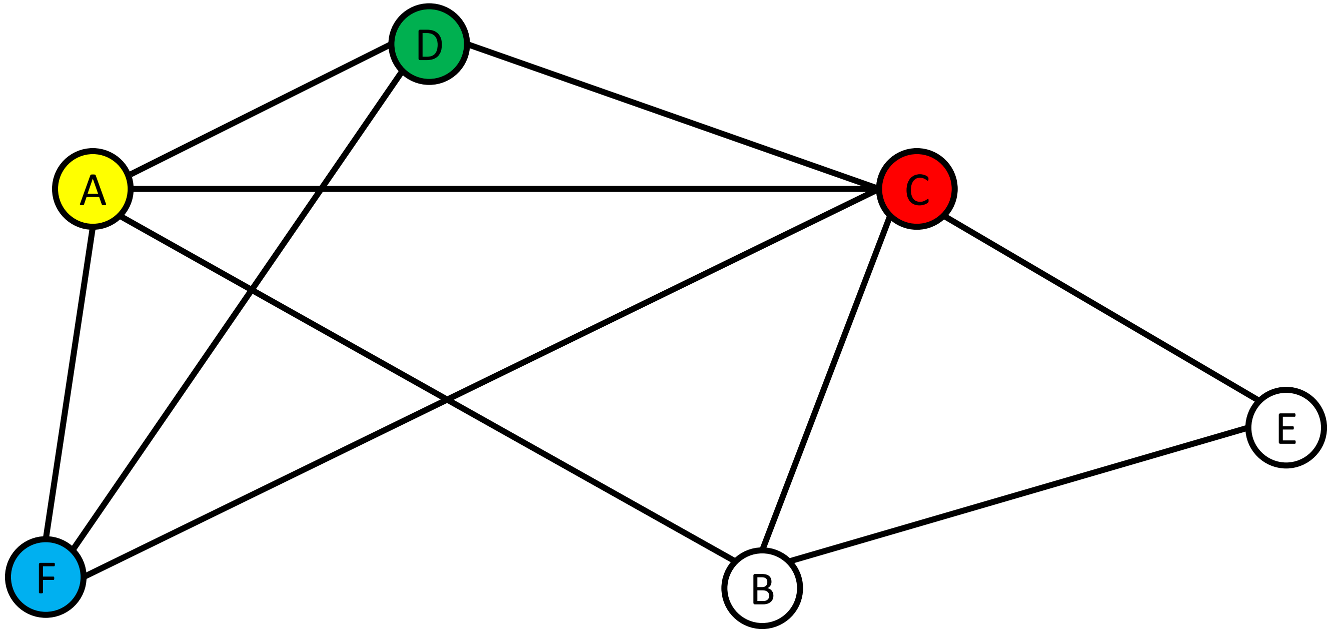
Summit	C	A	D	F	B	E
Degree	5	4	3	3	3	2
Color	Red	Yel				
saturation degree	1	1	2	2	2	1

DSatur Algorithm



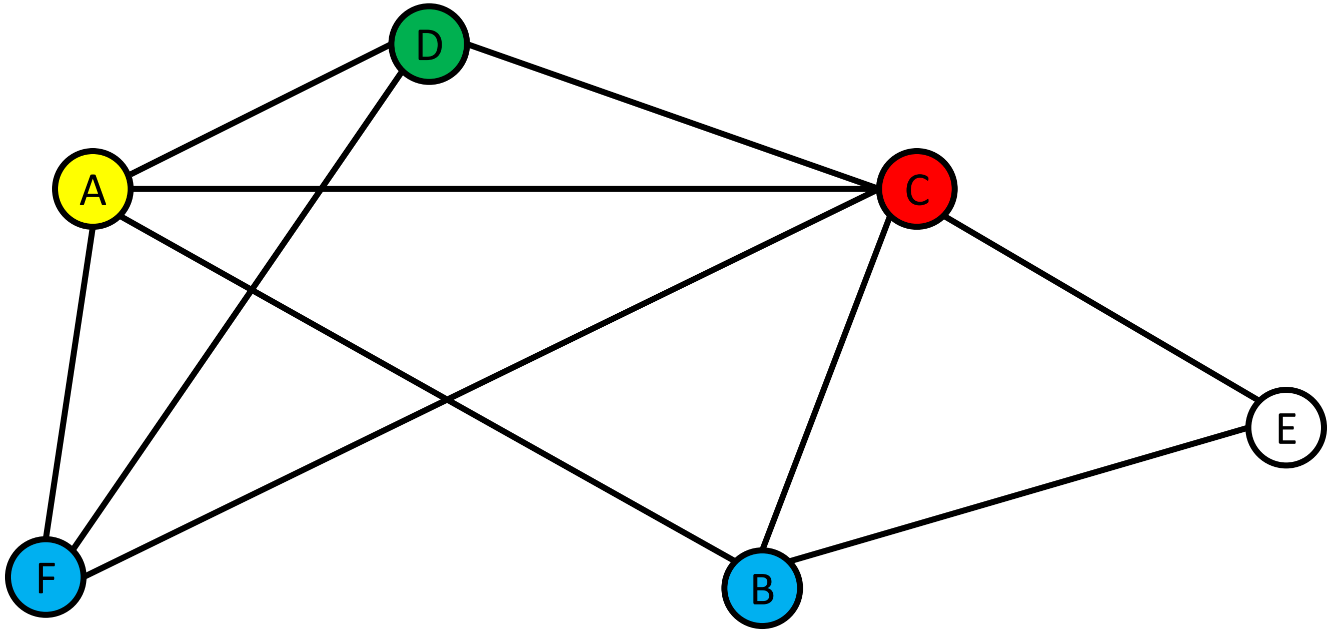
Summit	C	A	D	F	B	E
Degree	5	4	3	3	3	2
Color	Red	Yel	Gr			
saturation degree	2	2	2	3	2	1

DSatur Algorithm

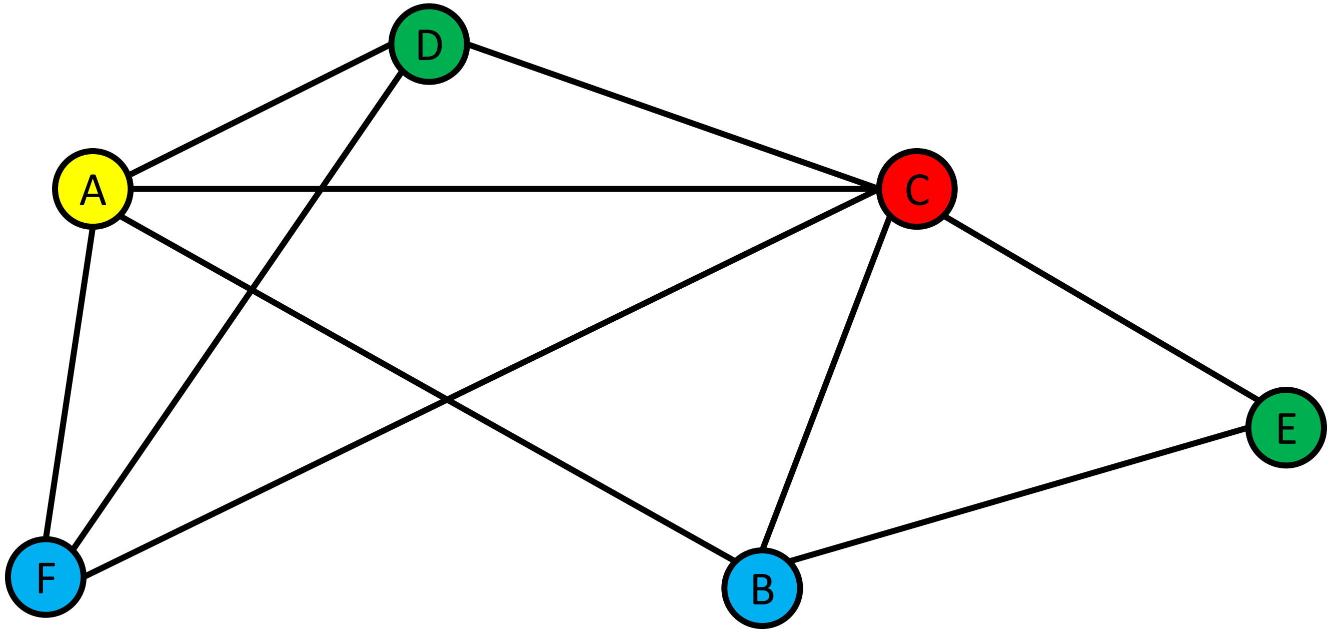


Summit	C	A	D	F	B	E
Degree	5	4	3	3	3	2
Color	Red	Yel	Gr	Blue		
saturation degree	3	3	3	3	2	1

DSatur Algorithm



Summit	C	A	D	F	B	E
Degree	5	4	3	3	3	2
Color	Red	Yel	Gr	Blue	Blue	
saturation degree	4	4	3	3	2	2

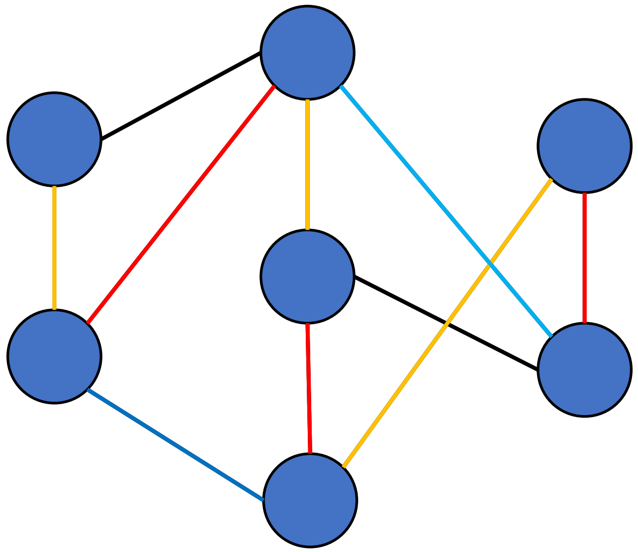


Summit	C	A	D	F	B	E
Degree	5	4	3	3	3	2
Color	Red	Yel	Gr	Blue	Blue	Gr
saturation degree	4	4	3	3	2	2

- Applying the coloring algorithm, it took four colors to color this graph.
- Since (A, C, D, F) forms a complete subgraph of order 4,
- the chromatic number of this graph is 4.
- It will therefore take 2 days to organize this exam.

- **Vertex coloring:** A proper vertex coloring problem for a given graph G is to color all the vertices of the graph with different colors in such a way that any two adjacent (having an edge connecting them) vertices of G have assigned different colors.
- **Edge coloring :** the edge coloring of a graph G = (V, E) is a mapping, which assigns a color to every edge, satisfying condition that no two edges sharing a common vertex have the same color.

Graph Edge Coloring



Chromatic Index = min
colors needed

Vizing's Theorem: every simple undirected graph may be edge colored using a number of colors that is at most one larger than the maximum degree Δ of the graph.

- "class one" graphs for which Δ colors suffice ,
- "class two" graphs for which Δ + 1 colors are necessary.