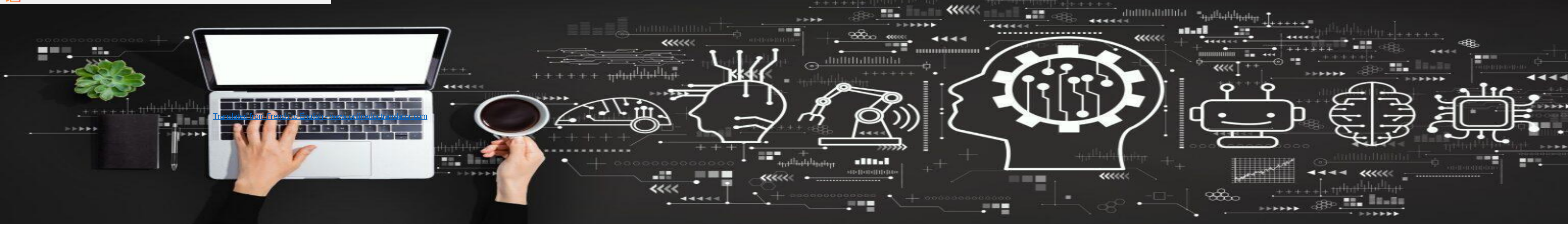

Numerical Methods

chapter 4:
Solving Linear Systems
Indirect Methods

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Solving Linear Systems

Indirect methods
(Iterative methods)

- **The Jacobi Method**

  Gauss-Seidel Method

Introduction

- Direct methods are much more suitable for solving small and medium-sized linear systems (in terms of number of equations and unknowns), their great advantage is that they give an exact solution (with or without rounding error) in a fast and direct way.
- Iterative methods aim to compute a sequence of solutions converging to the exact (final) solution. This type of method is very suitable for large systems even with a lot of floating point calculations.

Jacobi Iteration Method

This method makes two assumptions: (1) that the system

$$\begin{array}{ccccccccc}
a_{11}x_1 & + & a_{12}x_2 & + & \cdots & + & a_{1n}x_n & = & b_1 \\
a_{21}x_1 & + & a_{22}x_2 & + & \cdots & + & a_{2n}x_n & = & b_2 \\
\vdots & & \vdots & & & & \vdots & & \vdots \\
a_{n1}x_1 & + & a_{n2}x_2 & + & \cdots & + & a_{nn}x_n & = & b_n
\end{array}$$

Has a unique solution and (2) that the coefficient matrix A has no zeros on its main diagonal.

If any of the diagonal entries $a_{11}, a_{22}, \ldots, a_{nn}$ ro, then rows or columns must be interchanged to obtain a coefficient matrix that has all nonzero entries on the main diagonal.

To begin the Jacobi method, solve the first equation for x_1, the second equation for x_2, and so on, as follows

$$\begin{aligned}
x_1 &= \frac{1}{a_{11}}(b_1 - a_{12}x_2 - a_{13}x_3 - \cdots - a_{1n}x_n) \\
x_2 &= \frac{1}{a_{22}}(b_2 - a_{21}x_1 - a_{23}x_3 - \cdots - a_{2n}x_n) \\
&\vdots \\
x_n &= \frac{1}{a_{nn}}(b_n - a_{n1}x_1 - a_{n2}x_2 - \cdots - a_{n,n-1}x_{n-1})
\end{aligned}$$

Then make an *initial approximation* of the solution

$(x_1, x_2, x_3, \ldots, x_n)$ Initial approximation

Jacobi Iteration Method

Example: Use the Jacobi method to approximate the solution of the following system of linear equations.

$$\begin{aligned} 5x_1 - 2x_2 + 3x_3 &= -1 \\ -3x_1 + 9x_2 + x_3 &= 2 \\ 2x_1 - x_2 - 7x_3 &= 3 \end{aligned}$$

SOLUTION

To begin, write the system in the form

$$\begin{aligned} x_1 &= -\tfrac{1}{5} + \tfrac{2}{5}x_2 - \tfrac{3}{5}x_3 \\ x_2 &= \tfrac{2}{9} + \tfrac{3}{9}x_1 - \tfrac{1}{9}x_3 \\ x_3 &= -\tfrac{3}{7} + \tfrac{2}{7}x_1 - \tfrac{1}{7}x_2. \end{aligned}$$

Because you do not know the actual solution, choose

$$x_1 = 0, \quad x_2 = 0, \quad x_3 = 0 \qquad \text{Initial approximation}$$

as a convenient initial approximation. So, the first approximation is

$$\begin{aligned} x_1 &= -\tfrac{1}{5} + \tfrac{2}{5}(0) - \tfrac{3}{5}(0) = -0.200 \\ x_2 &= \tfrac{2}{9} + \tfrac{3}{9}(0) - \tfrac{1}{9}(0) \approx 0.222 \\ x_3 &= -\tfrac{3}{7} + \tfrac{2}{7}(0) - \tfrac{1}{7}(0) \approx -0.429. \end{aligned}$$

Now substitute these x-values into the system to obtain the second approximation.

$$\begin{aligned} x_1 &= -\tfrac{1}{5} + \tfrac{2}{5}(0.222) - \tfrac{3}{5}(-0.429) \approx 0.146 \\ x_2 &= \tfrac{2}{9} + \tfrac{3}{9}(-0.200) - \tfrac{1}{9}(-0.429) \approx 0.203 \\ x_3 &= -\tfrac{3}{7} + \tfrac{2}{7}(-0.200) - \tfrac{1}{7}(0.222) \approx -0.517. \end{aligned}$$

Continuing this procedure, you obtain the sequence of approximations shown in the table.

n	0	1	2	3	4	5	6	7
x_1	0.000	−0.200	0.146	0.191	0.181	0.185	0.186	0.186
x_2	0.000	0.222	0.203	0.328	0.332	0.329	0.331	0.331
x_3	0.000	−0.429	−0.517	−0.416	−0.421	−0.424	−0.423	−0.423

Because the last two columns in the table are identical, you can conclude that to three significant digits the solution is

$$x_1 = 0.186, \quad x_2 = 0.331, \quad x_3 = -0.423.$$

- **Stopping test:**

The recurrent calculation of this sequence must be repeated until:

- A given number of iterations k,
- or: $\boldsymbol{Max\ |x_i^{(k+1)} - x_i^{(k)}| < \varepsilon}$ith $\boldsymbol{\varepsilon > 0\ et\ 1 \le i \le n}$).

- **Conditions for convergence:**

The calculated sequence is convergent if this condition is satisfied:

- $\boldsymbol{|a_{ii}| > \sum_{\substack{j=1 \\ j \neq i}}^{n} |a_{ij}|\ i = 1, 2, .., n.}$ (diagonally dominant matrix)

Jacobi Algorithm

- $X^{(0)}$ given starting solution.

$$X_i^{(k+1)} = \frac{1}{a_{ii}} \left(b_i - \sum_{\substack{j=1 \\ j \neq i}}^{n} a_{ij} X_j^{(k)}\right) \quad i = 1, .., n$$

$$\begin{cases} X^{(0)} & \text{Given} \\ X_i^{(k+1)} = \dfrac{b_i - \sum_{j \neq i} a_{ij} X_j^{(k)}}{a_{ii}} & \forall i = 1, \ldots, n. \end{cases}$$

The Gauss-Seidel Method

- You will now look at a modification of the Jacobi method called the Gauss-Seidel method. This modification is no more difficult to use than the Jacobi method, and it often requires fewer iterations to produce the same degree of accuracy.

With the Jacobi method, the value x_i of obtained in the nth approximation remain unchanged until the entire (n+1)th approximation has been calculated. On the other hand, with the Gauss-Seidel method you use the new values of each x_i as soon as they are known. That is, once you have determined from x_1 the first equation, its value is then mused in the second equation to obtain the new x_2. Similarly, the new and are used in the third equation to obtain the new x_3, and so on. This procedure is demonstrated in Example below.

The Gauss-Seidel Method

Example: Use the Gauss-Seidel iteration method to approximate the solution of the following system of linear equations.

$$\begin{aligned} 5x_1 - 2x_2 + 3x_3 &= -1 \\ -3x_1 + 9x_2 + x_3 &= 2 \\ 2x_1 - x_2 - 7x_3 &= 3 \end{aligned}$$

SOLUTION

The first computation is identical to that in Example 1. That is, using $(x_1, x_2, x_3) = (0, 0, 0)$ as the initial approximation, you obtain the new value of x_1.

$$x_1 = -\tfrac{1}{5} + \tfrac{2}{5}(0) - \tfrac{3}{5}(0) = -0.200$$

Now that you have a new value of x_1, use it to compute a new value of x_2. That is,

$$x_2 = \tfrac{2}{9} + \tfrac{3}{9}(-0.200) - \tfrac{1}{9}(0) \approx 0.156.$$

Similarly, use $x_1 = -0.200$ and $x_2 = 0.156$ to compute a new value of x_3. That is,

$$x_3 = -\tfrac{3}{7} + \tfrac{2}{7}(-0.200) - \tfrac{1}{7}(0.156) = -0.508.$$

So, the first approximation is $x_1 = -0.200$, $x_2 = 0.156$, and $x_3 = -0.508$. Now performing the next iteration produces

$$\begin{aligned} x_1 &= -\tfrac{1}{5} + \tfrac{2}{5}(0.156) - \tfrac{3}{5}(-0.508) \approx 0.167 \\ x_2 &= \tfrac{2}{9} + \tfrac{3}{9}(0.167) - \tfrac{1}{9}(-0.508) \approx 0.334 \\ x_3 &= -\tfrac{3}{7} + \tfrac{2}{7}(0.167) - \tfrac{1}{7}(0.334) \approx -0.429. \end{aligned}$$

Continued iterations produce the sequence of approximations shown in the table.

n	0	1	2	3	4	5	6
x_1	0.000	−0.200	0.167	0.191	0.187	0.186	0.186
x_2	0.000	0.156	0.334	0.334	0.331	0.331	0.331
x_3	0.000	−0.508	−0.429	−0.422	−0.422	−0.423	−0.423

$$x_1 = 0.186, \quad x_2 = 0.331, \quad x_3 = -0.423.$$

Gauss-Seidel Algorithm

- $X^{(0)}$ given starting solution.

$$X_i^{(k+1)} = \frac{1}{aii}\left(b_i - \sum_{\substack{j=1\\ j\neq i}}^{n} a_{ij}X_j^{(k)}\right) \quad i = 1, ..,n$$

$$\begin{cases} X^{(0)} & \text{Given} \\ X_i^{(k+1)} = \dfrac{b_i - \sum_{j<i} a_{ij}X_j^{(k+1)} - \sum_{j>i} a_{ij}X_j^{(k)}}{a_{ii}} & \forall i = 1, \ldots, n. \end{cases}$$