Graph theory

Flow problems

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Flows: definitions and properties 1

- Flow networks are a fundamental concept in graph theory used to model the movement of resources through interconnected nodes.
- A flow network is a directed graph where a certain homogeneous quantity can stream between connected vertices through edges.This quantity is called flow.
- Each edge is characterized by a capacity, a non-negative real number.
- There are also two special vertices:
- $s \in V$ is the **source** vertex from which the flow originates.
- $t \in V$ is the **sink** vertex where the flow is collected.
  - No arc reaches the source.
  - No arc leaves the sink.

Flows: definitions and properties 2

- A flow non-negative real-valued function *f* defined on the arcs satisfying:
  - *Capacity constraint* : $f(a) \leq c(a)$;
  - *Conservation of flow* : for any vertex other than *s* and *t* , the sum of the flows on the incoming edges and the sum of the flows on the outgoing edges are equal.
  - $f(x,y) = -f(y,x)$
- Examples: electrical or hydraulic circuits, communication networks, transport modeling

Flows: definitions and properties 3

- the sum of the flows on the arcs leaving the source and the sum of the flows on the arcs arriving at the sink are equal;
  - This value is the value of the flow | f | ;
- if we separate the vertices into two subsets **E** containing **s** and **F** = **A** − **E**, containing **t**, then the sum of the values of the flow on the arcs from **E** to **F** minus the sum of the values of the flow on the arcs from **F** to **E** is also | f |.
- Such a separation into two subsets of the vertices is called a cut and this difference in flow sums is called the net flow crossing the cut.

Ford and Fulkerson Algorithm

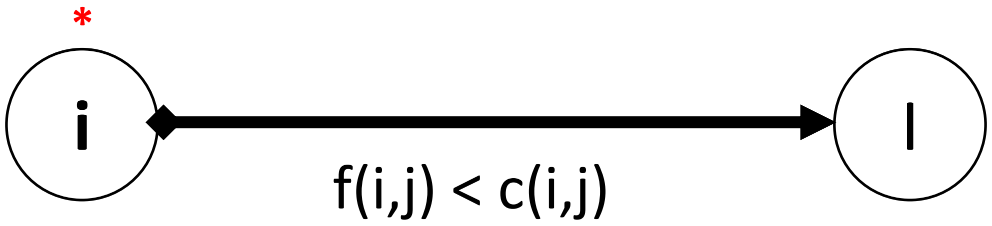
```
Max_Flow (G)
{
    MF=0; //the value of the max flow
    While (there exists an augmenting path in G) Do
    {
        Find a path increasing P
        C f (P) = smallest capacity on P
        MF=MF+ C f (P)
        For each arc ( u,v ) in P {
        c f ( u,v )= c f ( u,v ) - C f (P)
        c f ( seen )= c f ( seen ) + C f (P)
        }
    }
}
```

Definition: increasing path

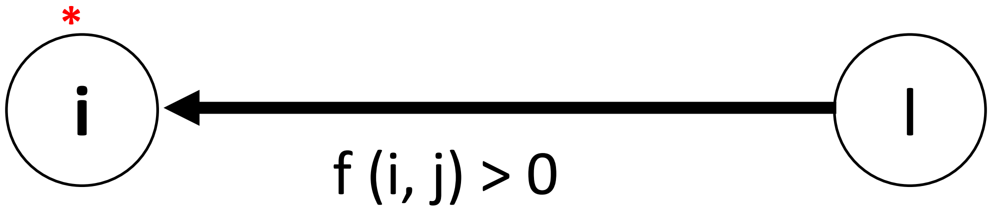
- A path C between "**s**" and "**t**" is said to be increasing with respect to a flow $f = (f(i,j), (i,j) \in E)$ feasible between **s** and **t** if:
  - $f(i,j) < c(i,j)$ if $(i,j) \in C^+$ ($(i,j) \in E$) conformal arc)
  - $f(i,j) > 0$ if $(i,j) \in C^-$ ($(j,i) \in E$) non-conforming arc)
- C^+ is the set of all arcs of C that meet in the right direction.
- C^- is the set of arcs of C that meet in the opposite direction.

Marking procedure

- Direct marking:
  - If for an arc (i, j) we have: i marked, j unmarked and f (i, j) < c(i, j) Then we mark j by (i,+) and we set $\delta(j) = \min\big(\delta(i), c(i,j) - f(i,j)\big)$

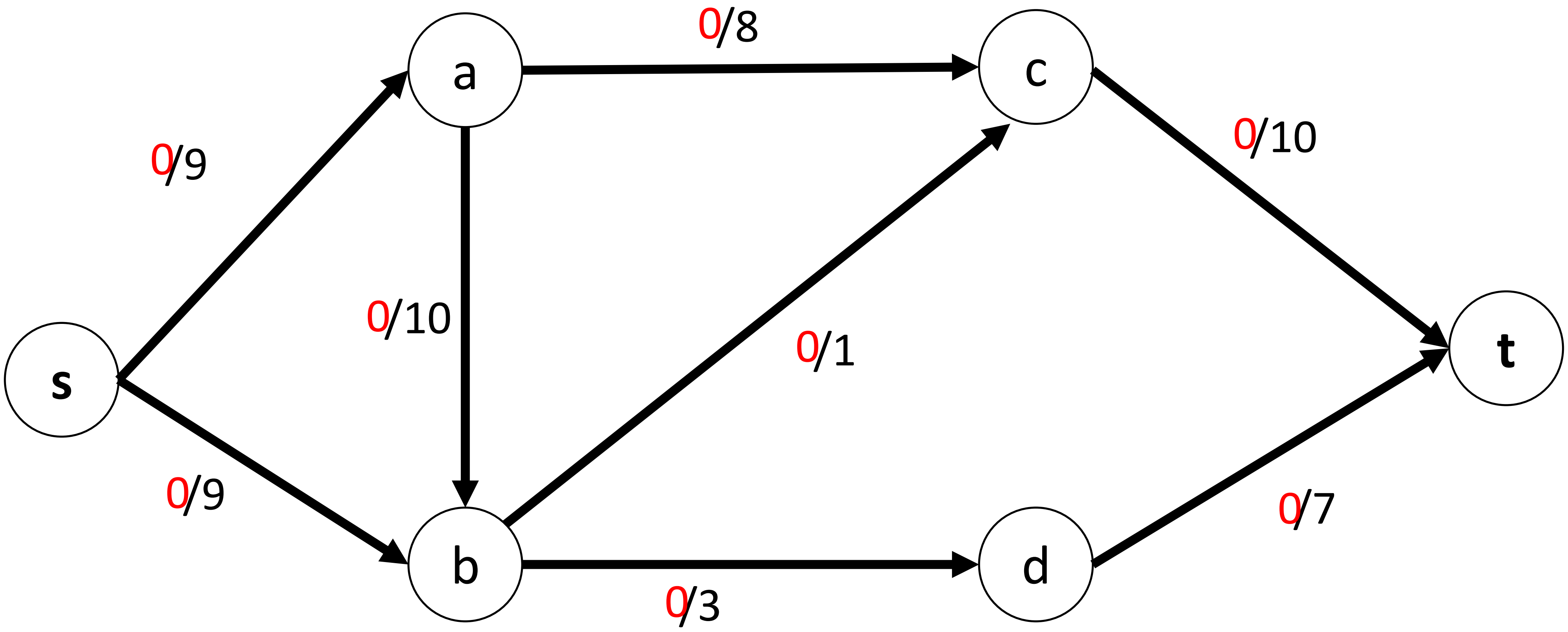


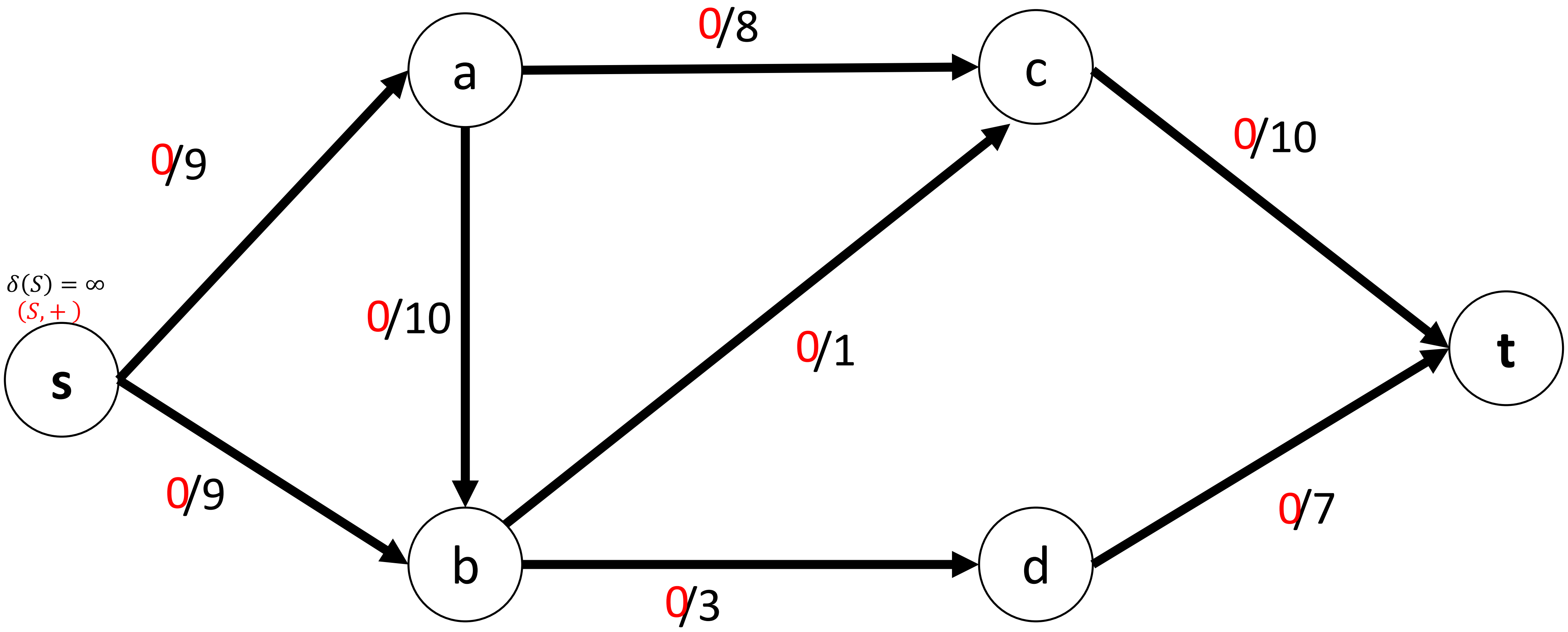
- Indirect marking:
  - If for an arc (j, i) we have: i marked, j unmarked and f (j, i) >0 Then we mark j by (i,-) and we set $\delta(j) = \min\big(\delta(i),\ \ f(j,i)\big)$



- $\delta(j)$ is the maximum flow you can increase from **s** to **j**.
- $\delta(i)$ is a value associated with vertex **i** , initialized to infinity for **s**.

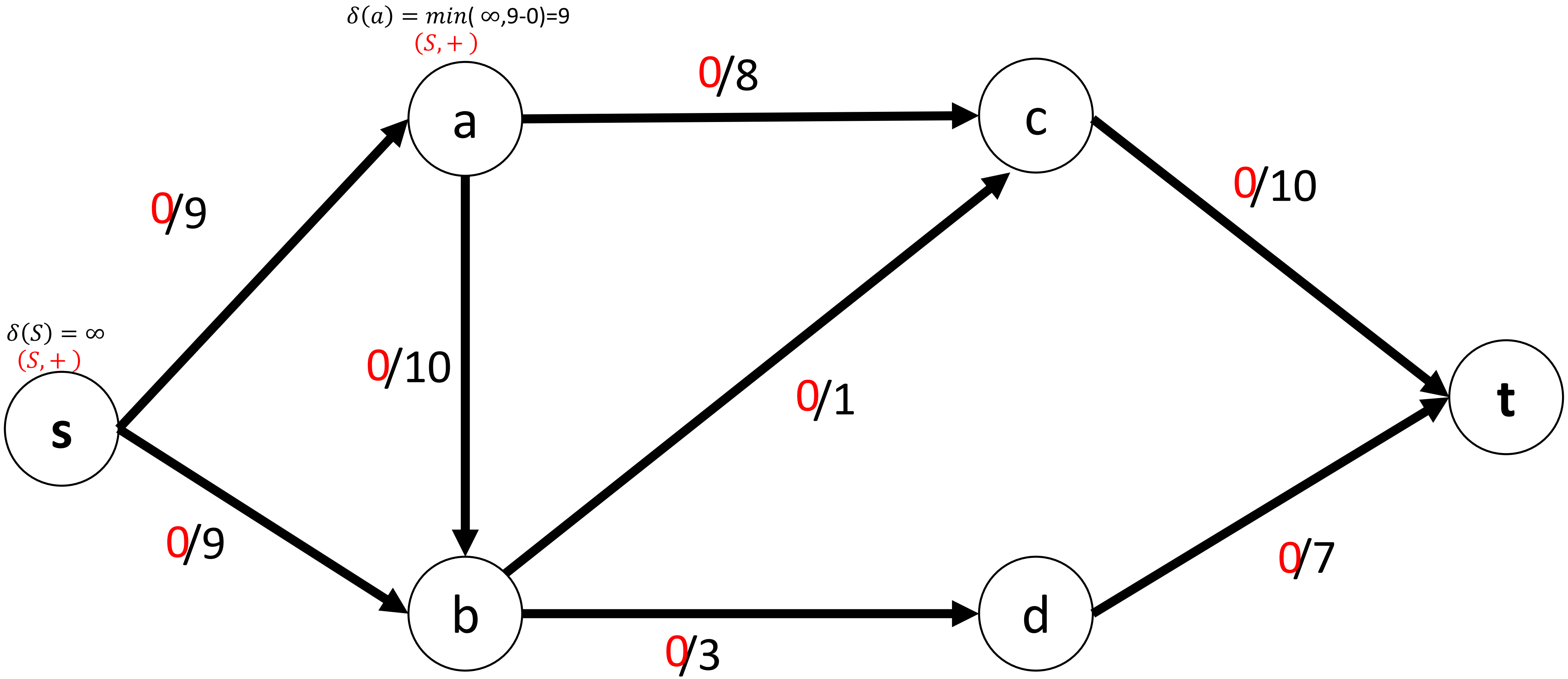
1

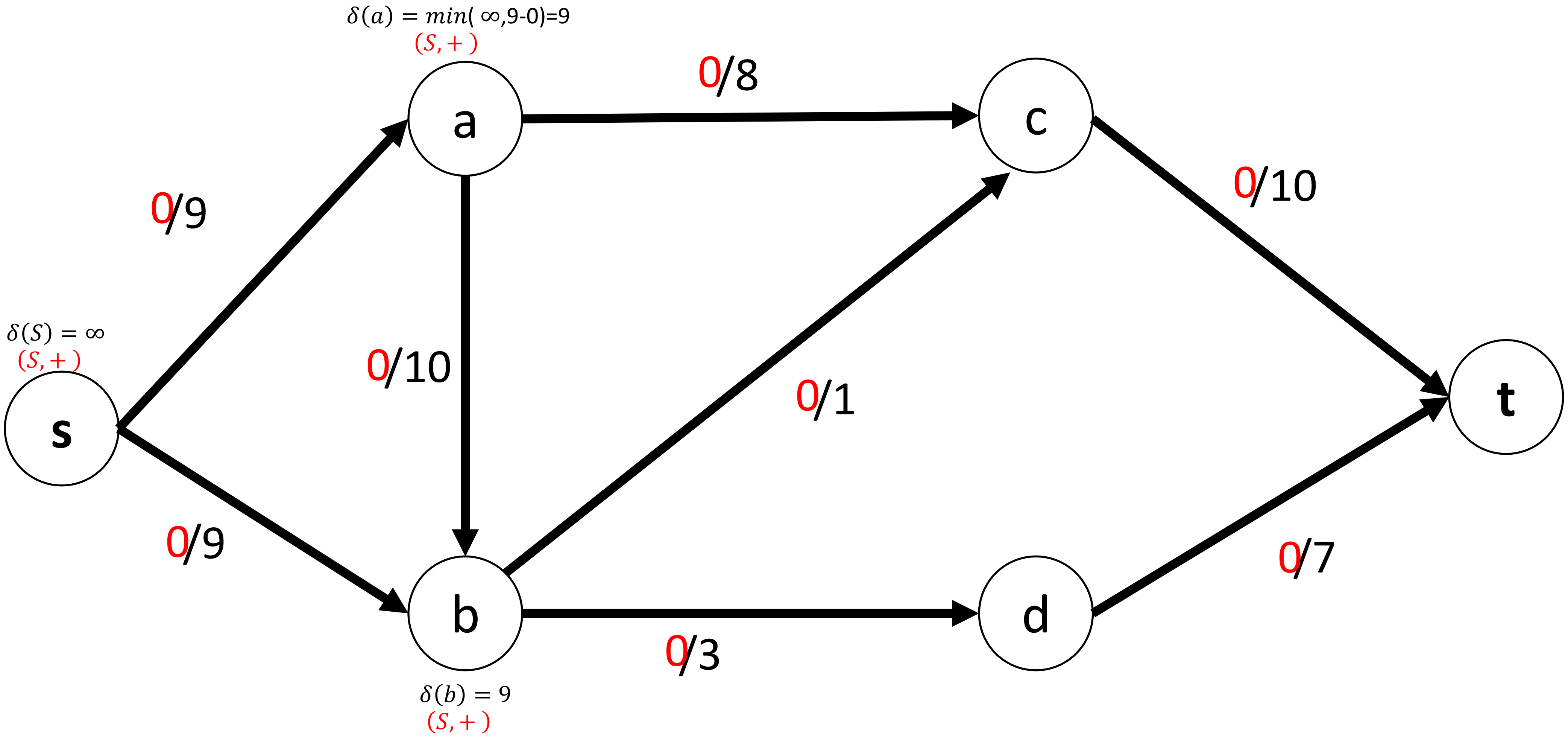




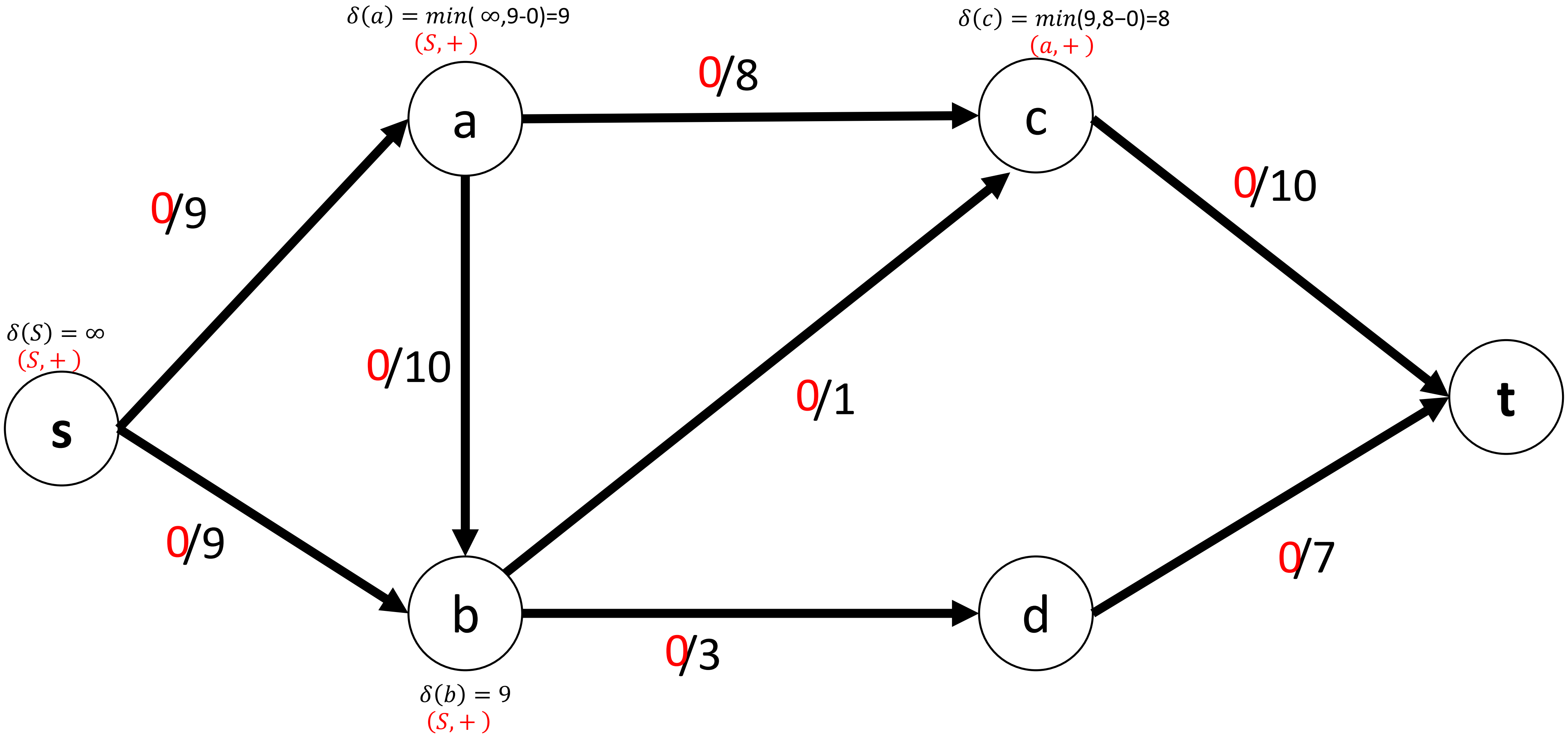
$\delta(S) = \infty$
$(S, +)$
s
a
b
c
d
t
0/9
0/9
0/10
0/8
0/1
0/3
0/10
0/7

1

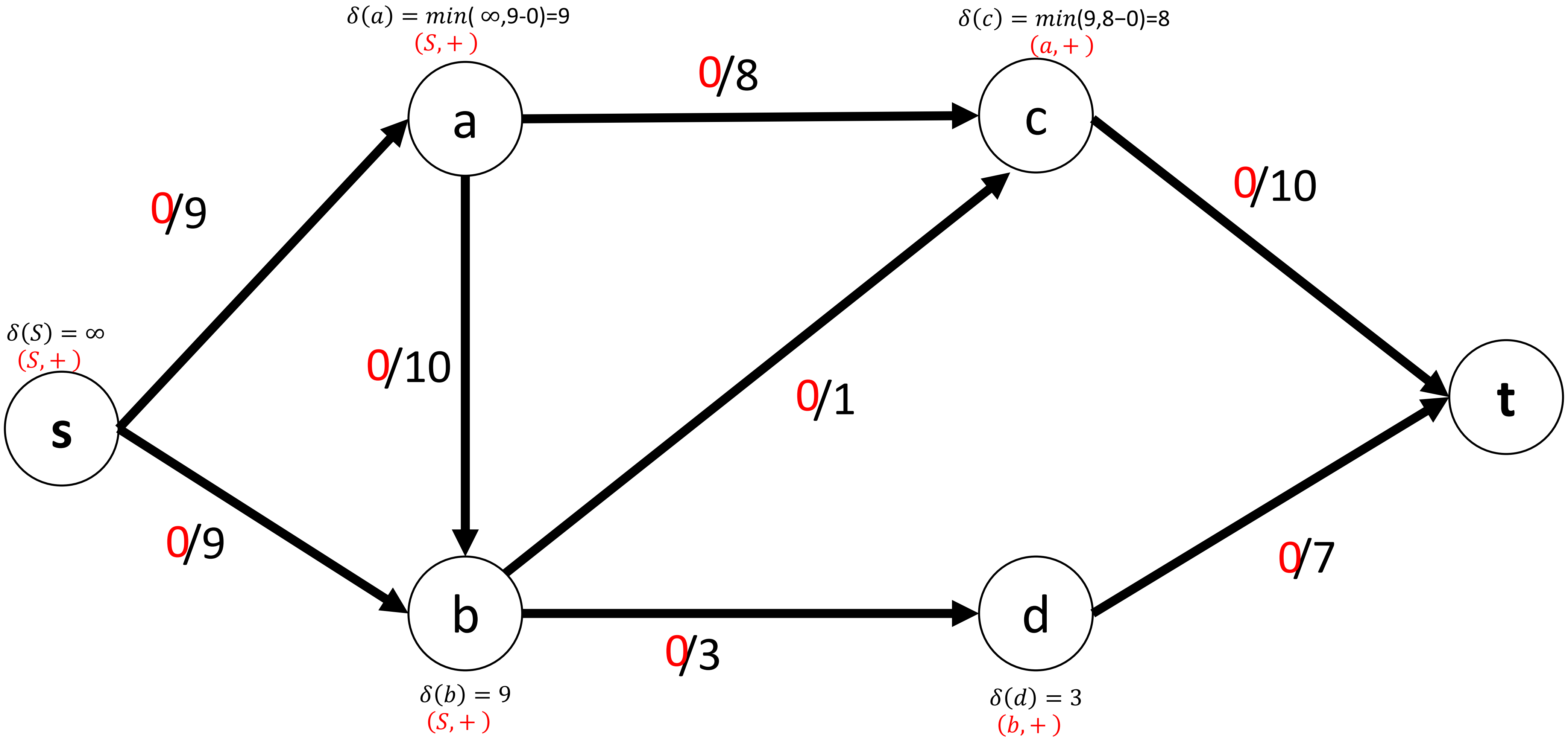




δ(a) = min(∞,9-0)=9
(S,+)
0/8
a
c
0/9
0/10
0/10
δ(S) = ∞
(S,+)
0/1
t
s
0/9
0/7
b
d
0/3
δ(b) = 9
(S,+)

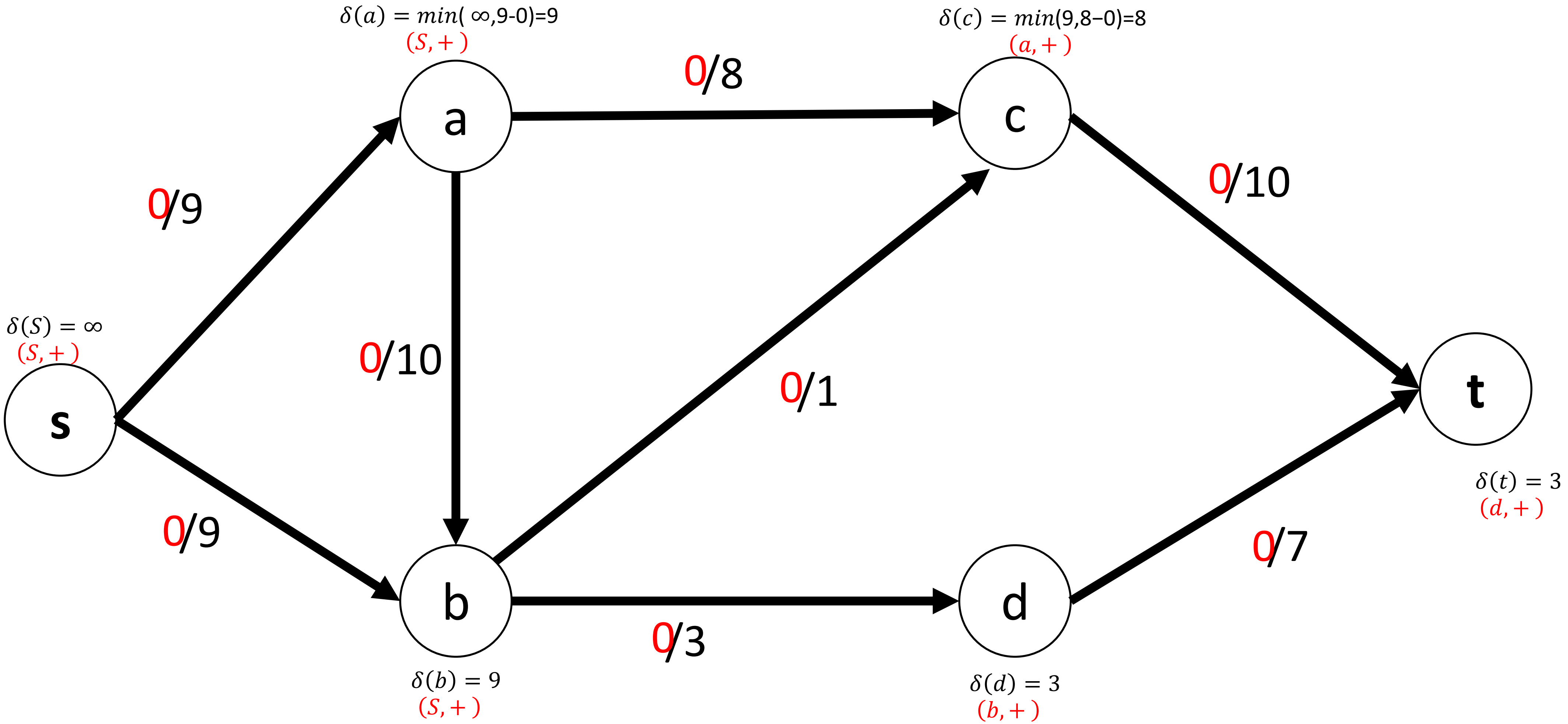

δ(a) = min(∞,9-0)=9
(S,+)
δ(c) = min(9,8−0)=8
(a,+)
0/8
a
c
0/9
0/10
0/10
δ(S) = ∞
(S,+)
s
0/1
t
0/9
0/7
b
d
0/3
δ(b) = 9
(S,+)

1



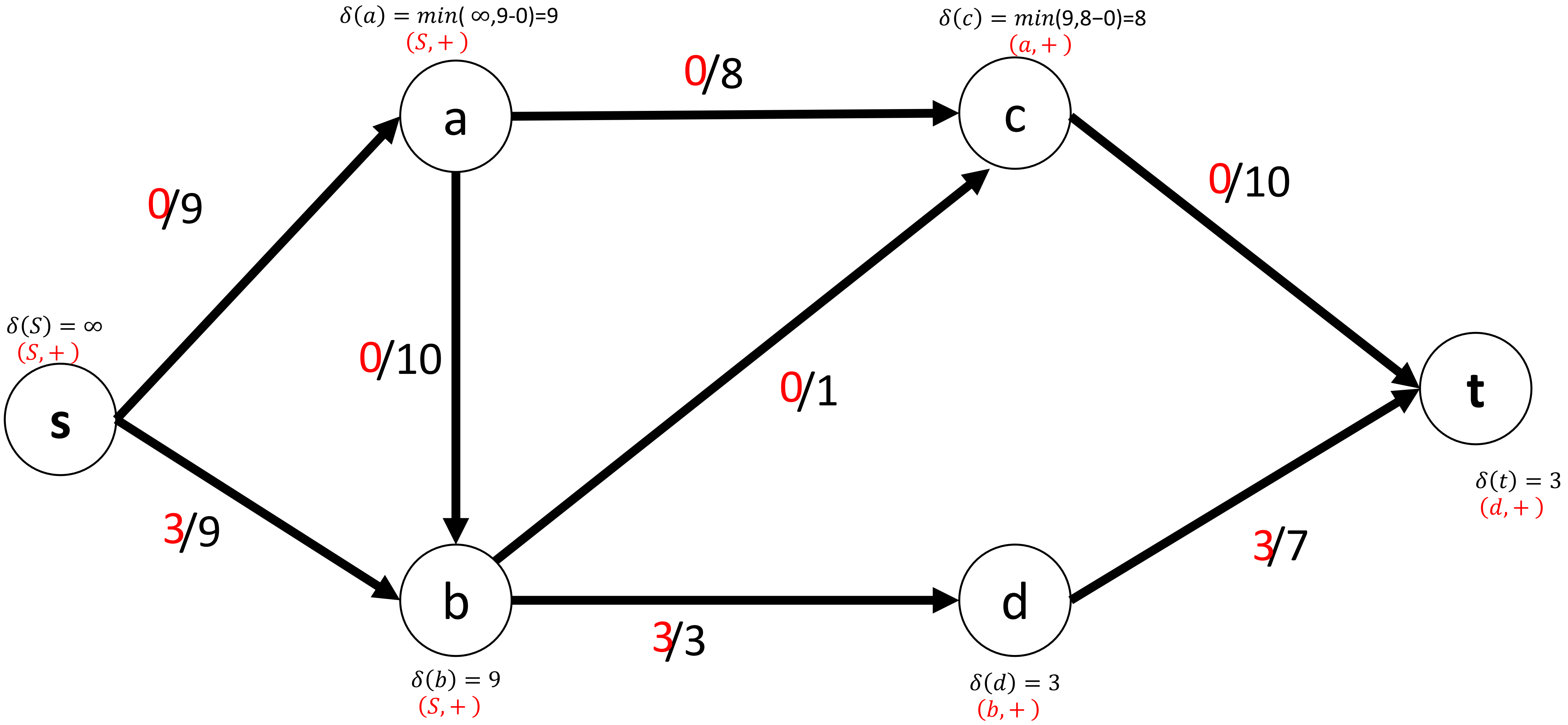
δ(a) = min(∞,9-0)=9
(S, +)
δ(c) = min(9,8−0)=8
(a, +)
0/8
a
c
0/9
0/10
δ(S) = ∞
(S, +)
0/10
0/1
s
t
0/9
0/7
b
d
0/3
δ(b) = 9
(S, +)
δ(d) = 3
(b, +)

1

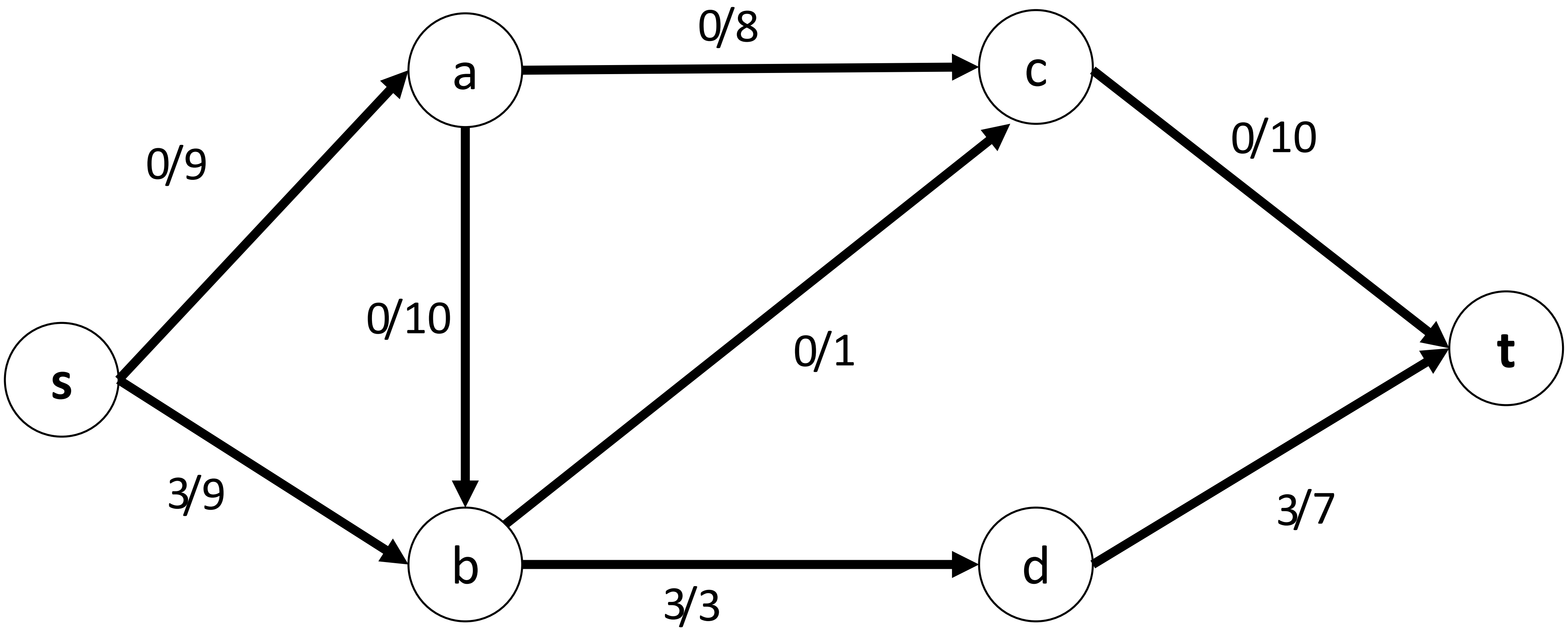


δ(a) = min(∞,9-0)=9
(S,+)
δ(c) = min(9,8−0)=8
(a,+)
0/8
a
c
0/9
0/10
δ(S) = ∞
(S,+)
0/10
0/1
s
t
0/9
δ(t) = 3
(d,+)
0/7
b
d
0/3
δ(b) = 9
(S,+)
δ(d) = 3
(b,+)

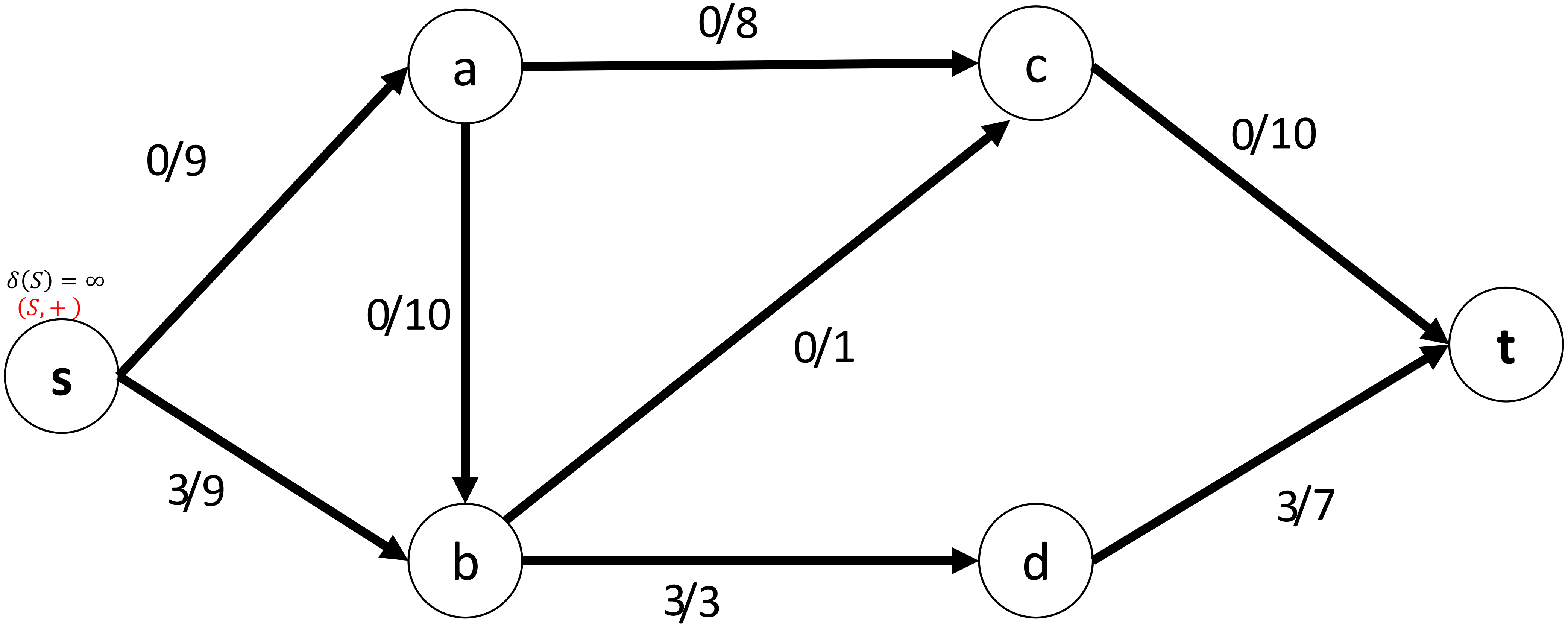
1

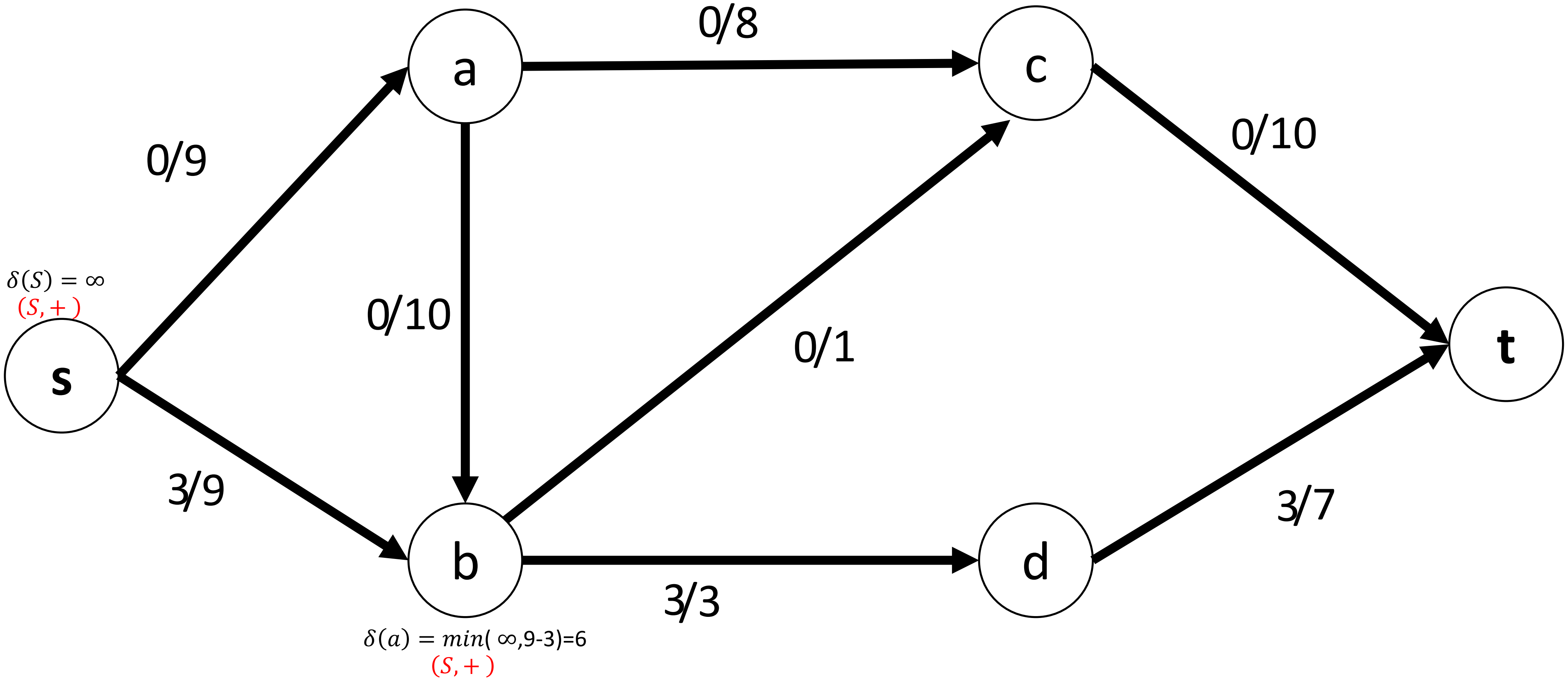


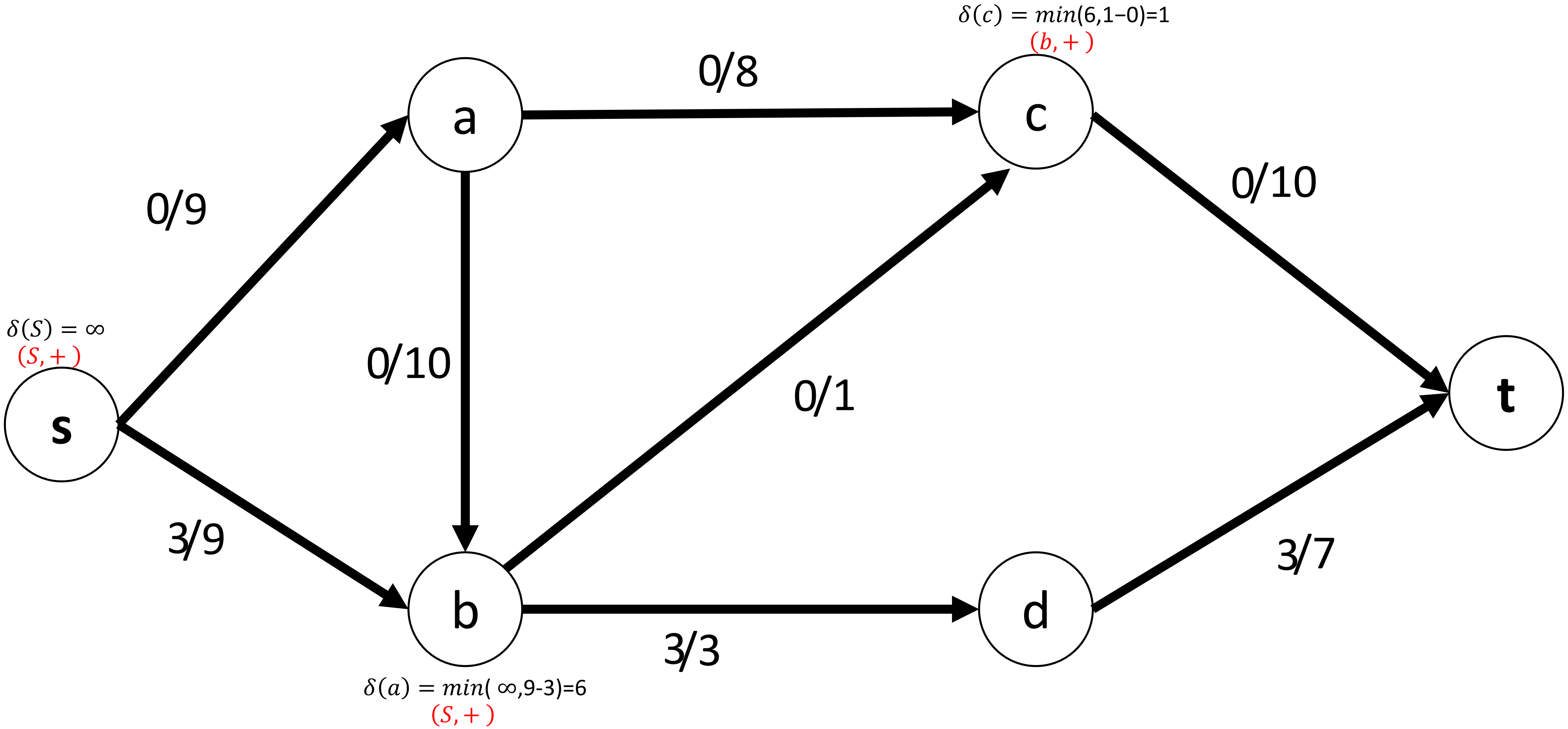
$\delta(a) = min(\infty$,9-0)=9
$(S,+)$
0/8
$\delta(c) = min$(9,8−0)=8
$(a,+)$
a
c
0/9
0/10
0/10
$\delta(S) = \infty$
$(S,+)$
s
0/1
t
$\delta(t) = 3$
$(d,+)$
3/9
3/7
b
d
3/3
$\delta(b) = 9$
$(S,+)$
$\delta(d) = 3$
$(b,+)$


0/8
a
c
0/10
0/9
0/10
0/1
s
t
3/9
3/7
b
d
3/3

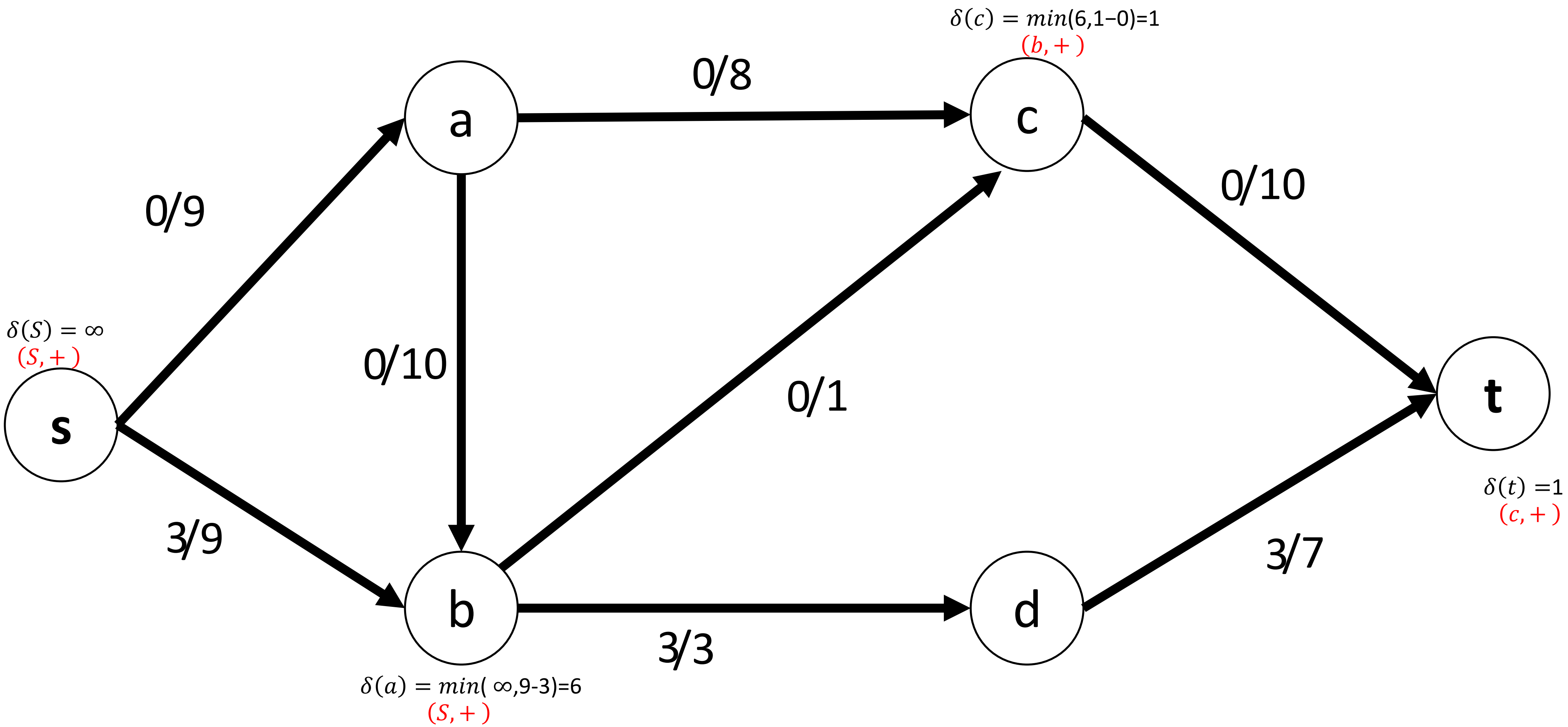
2



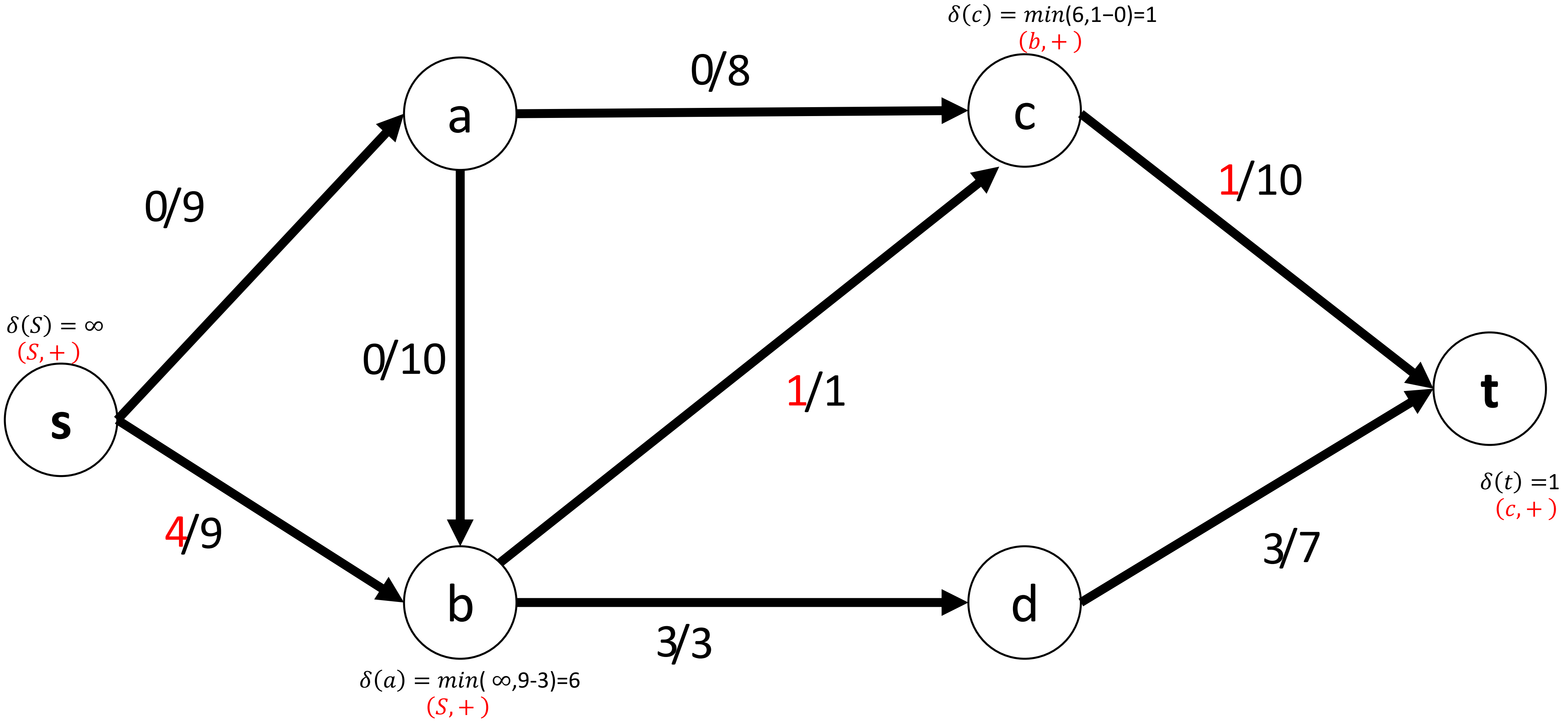

0/8
0/9
0/10
0/10
0/1
δ(S) = ∞
(S, +)
3/9
3/3
3/7
δ(a) = min(∞,9-3)=6
(S, +)
a
c
s
t
b
d

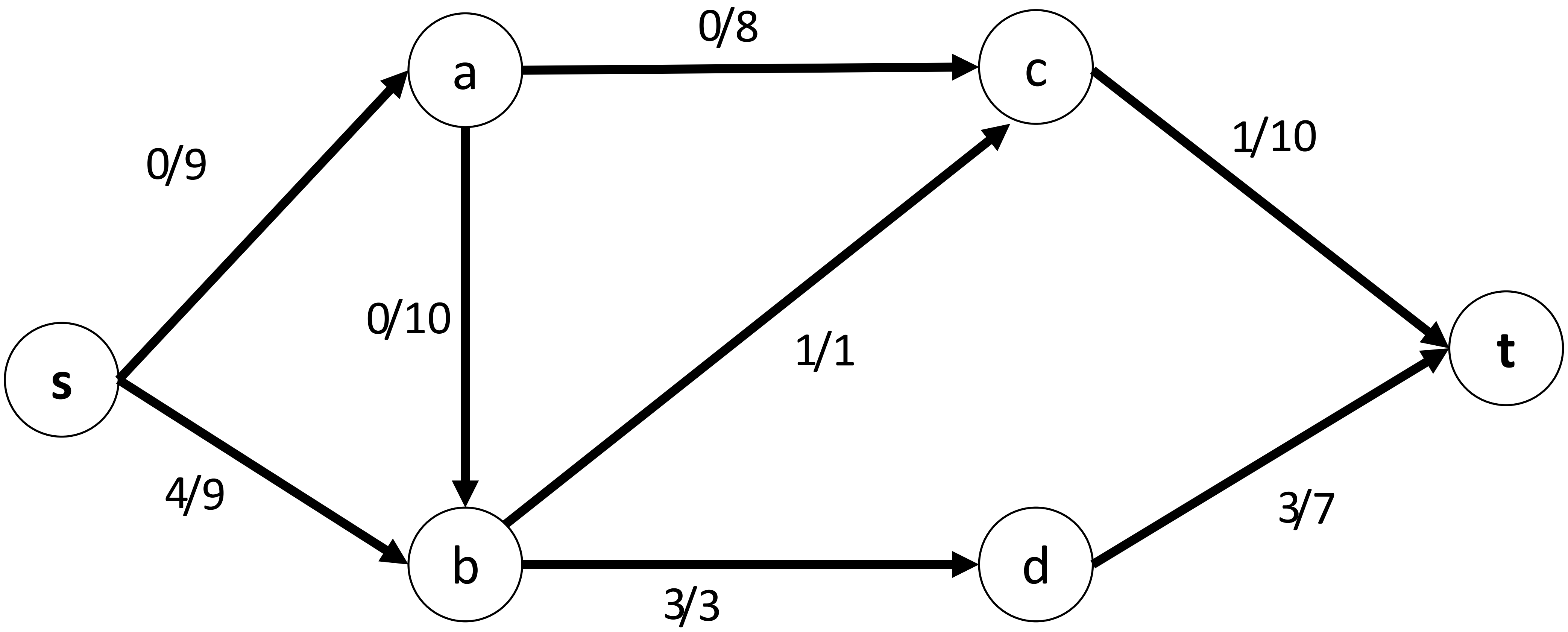


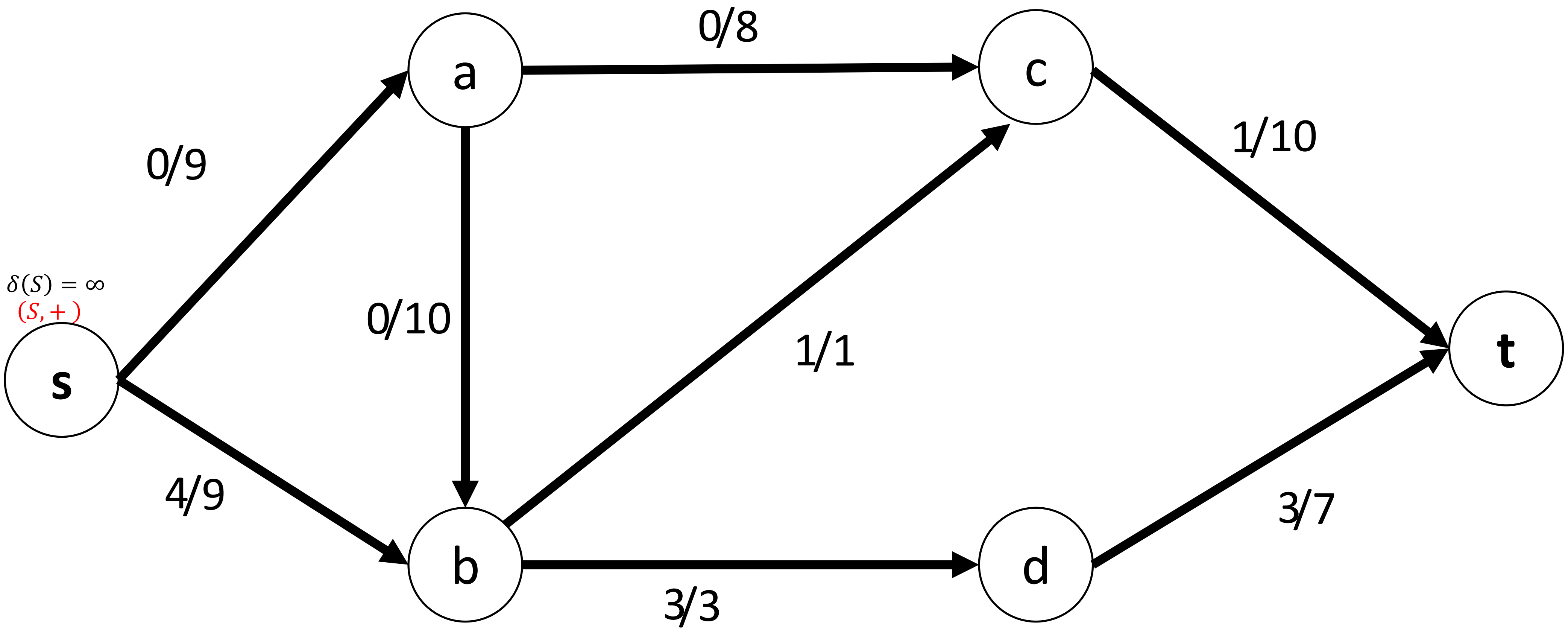
$\delta(c) = min$(6,1−0)=1
$(b, +)$
$\delta(S) = \infty$
$(S, +)$
a
c
s
t
b
d
0/8
0/9
0/10
0/10
0/1
3/9
3/7
3/3
$\delta(a) = min$(∞,9-3)=6
$(S, +)$


$\delta(c) = min(6,1-0)=1$
$(b,+)$
0/8
a
c
0/10
0/9
$\delta(S) = \infty$
$(S,+)$
0/10
0/1
s
t
$\delta(t) =1$
$(c,+)$
3/9
3/7
b
d
3/3
$\delta(a) = min(\infty,9-3)=6$
$(S,+)$

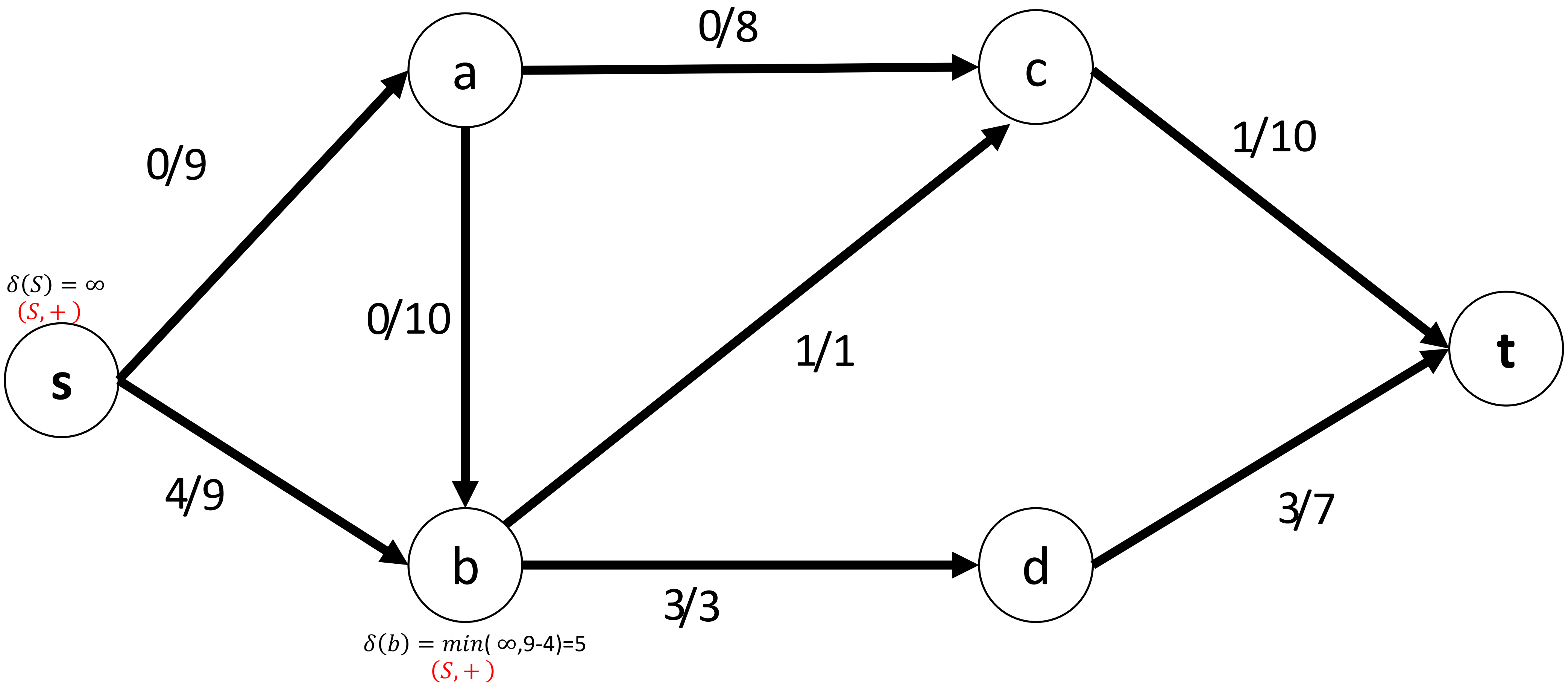
2


δ(c) = min(6,1−0)=1
(b, +)
0/8
a
c
0/9
1/10
δ(S) = ∞
(S, +)
0/10
1/1
s
t
δ(t) =1
(c, +)
4/9
3/7
b
d
3/3
δ(a) = min(∞,9-3)=6
(S, +)

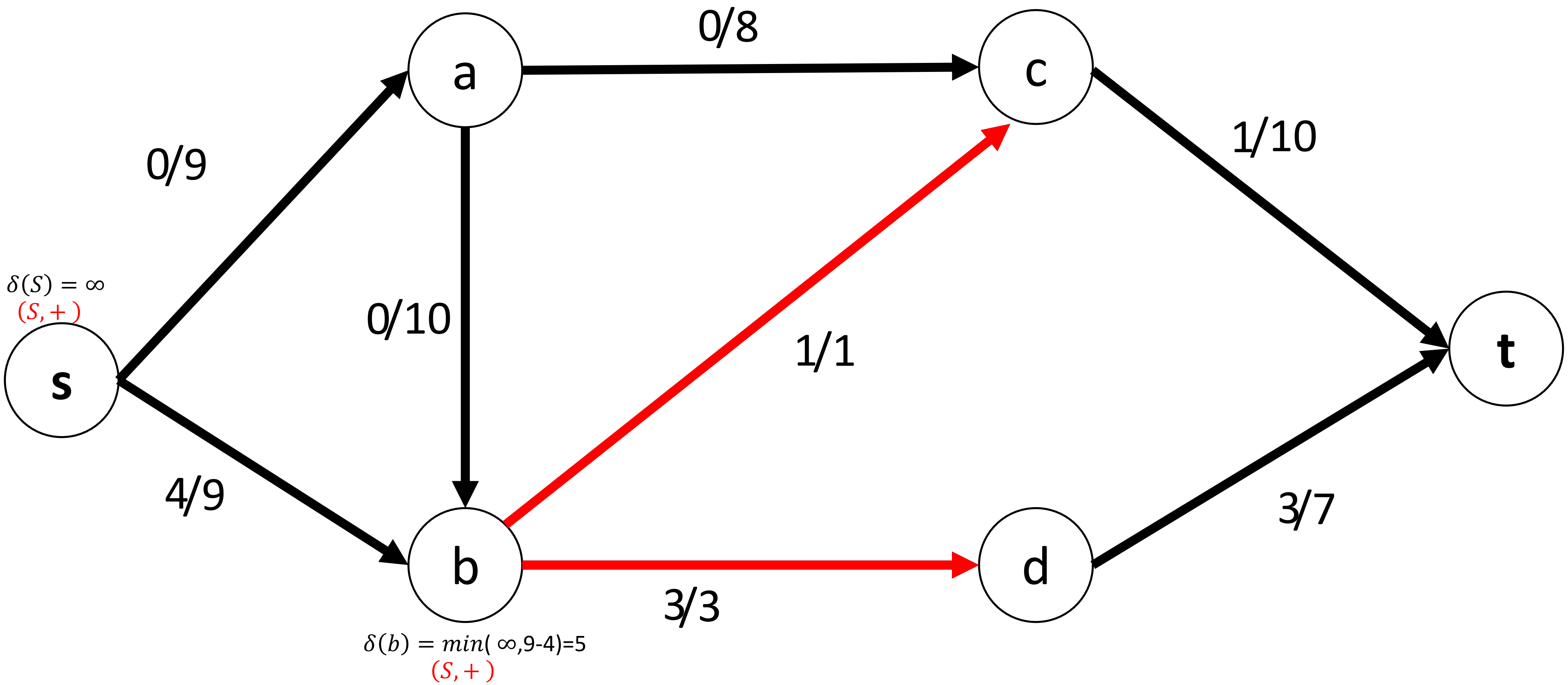

0/8
a
c
0/9
1/10
0/10
s
1/1
t
4/9
3/7
b
d
3/3



$\delta(S) = \infty$
$(S, +)$
s
a
b
c
d
t
0/9
4/9
0/8
0/10
1/1
3/3
1/10
3/7

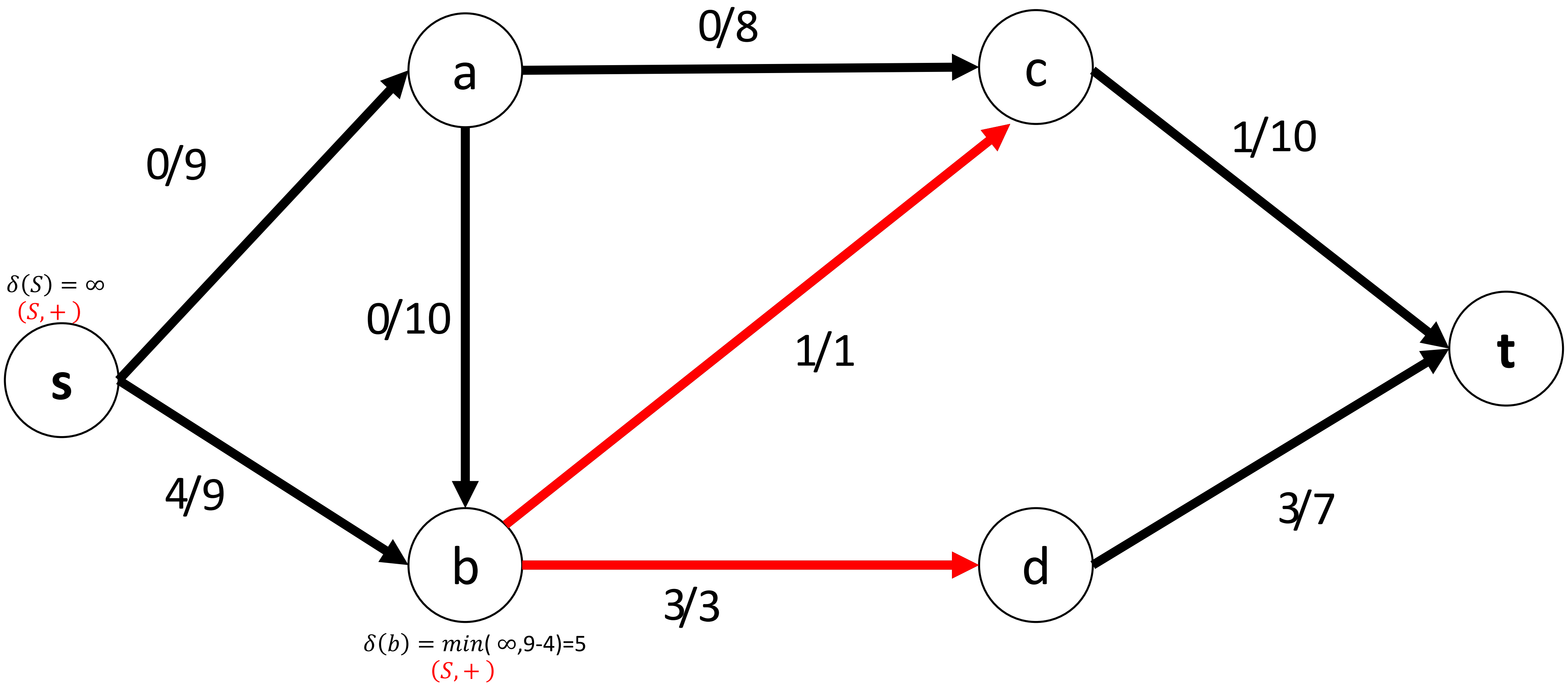

0/8
0/9
1/10
0/10
1/1
4/9
3/7
3/3
a
c
s
t
b
d
$\delta(S) = \infty$
$(S, +)$
$\delta(b) = min(\infty$,9-4)=5
$(S, +)$

3



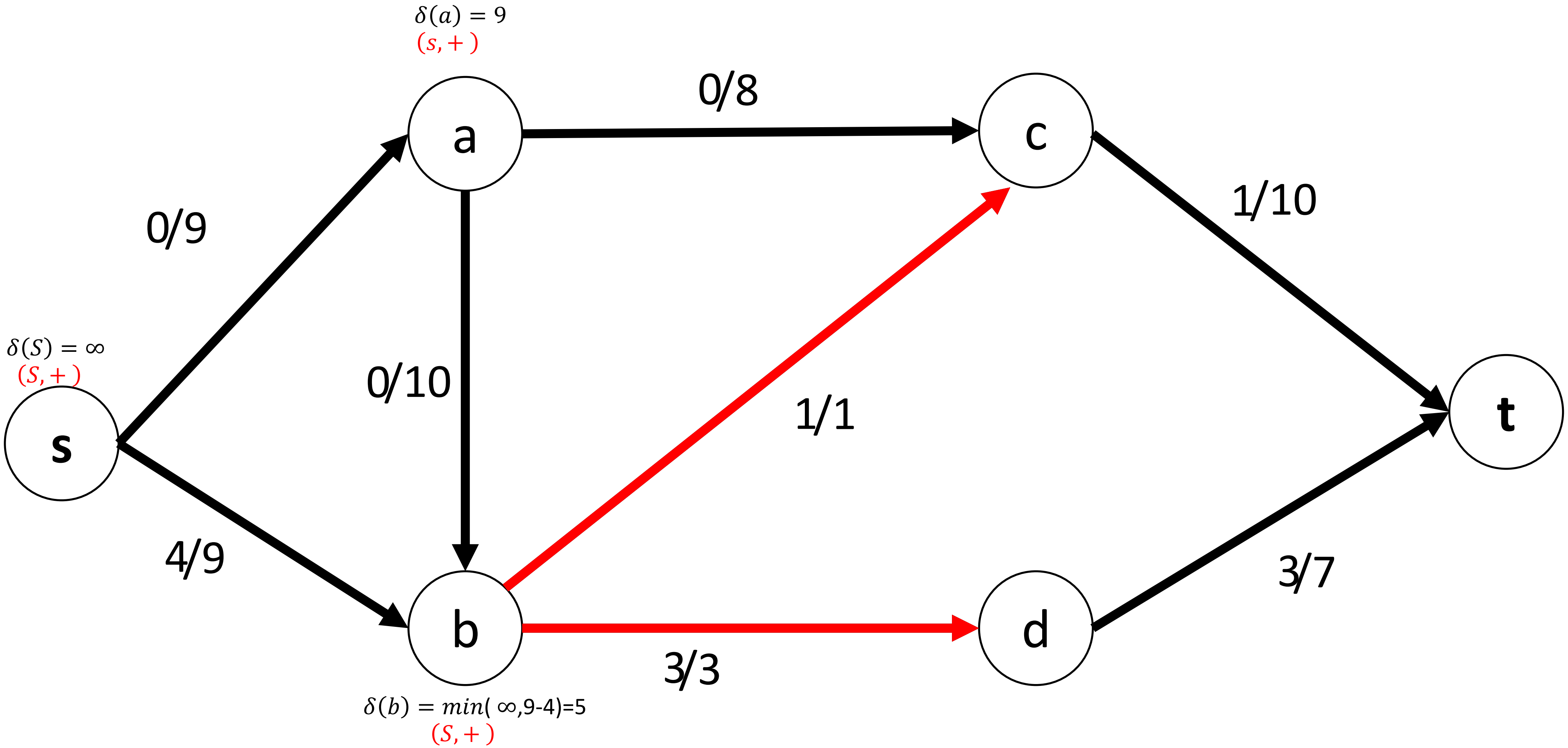
a
c
0/8
0/9
1/10
0/10
1/1
$\delta(S) = \infty$
$(S, +)$
s
t
4/9
3/7
b
d
3/3
$\delta(b) = min(\infty$,9-4)=5
$(S, +)$

3



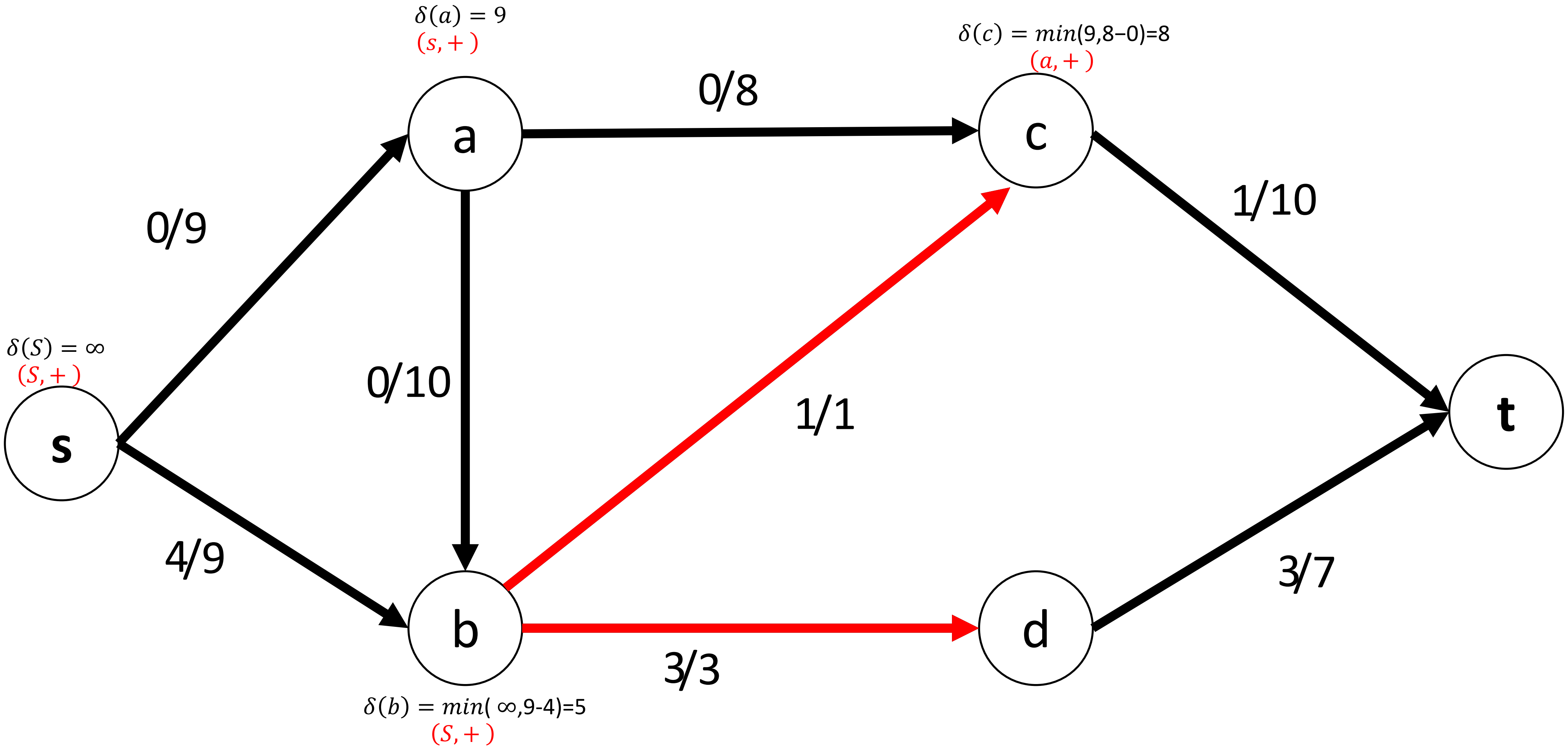
a
c
s
t
b
d
0/8
0/9
1/10
0/10
1/1
4/9
3/7
3/3
$\delta(S) = \infty$
$(S, +)$
$\delta(b) = min(\infty$,9-4)=5
$(S, +)$

3

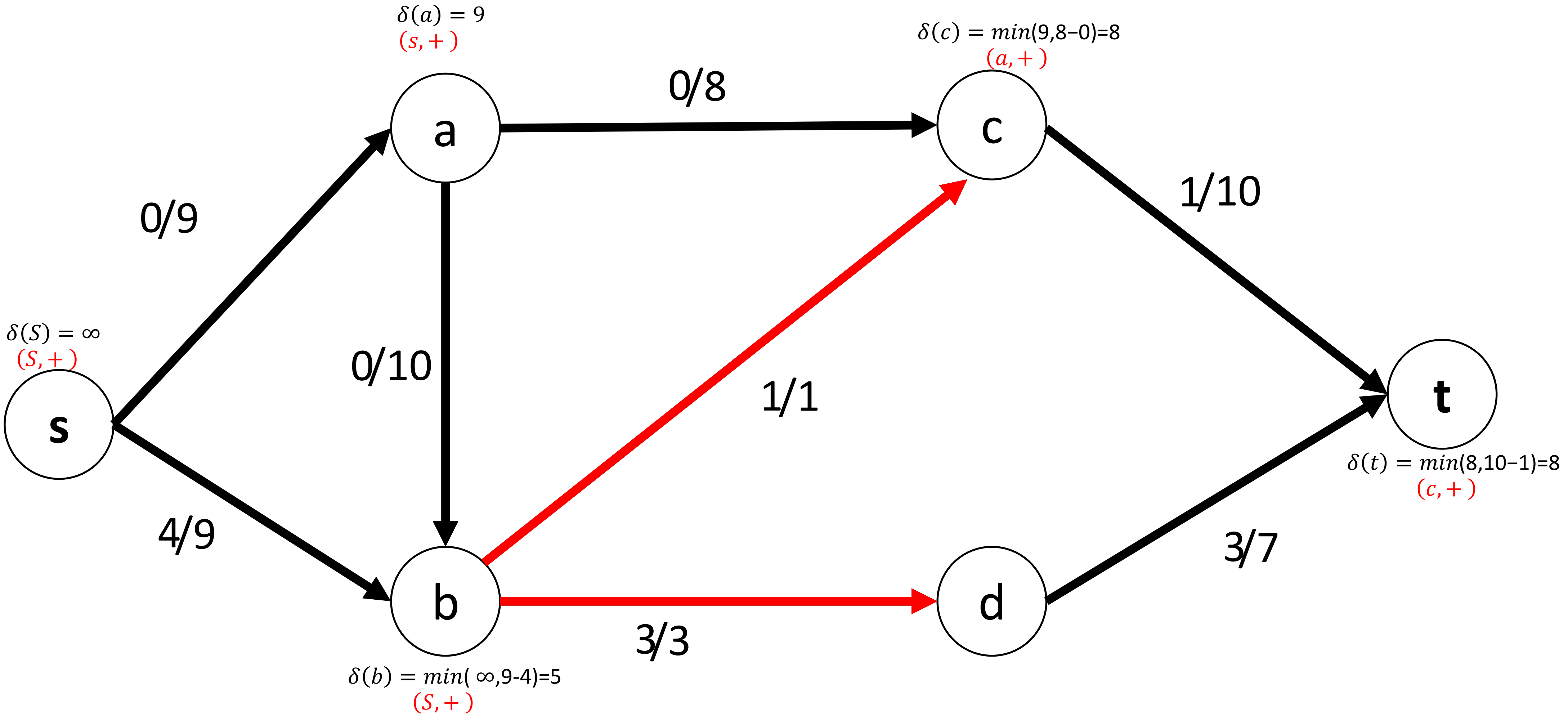


$\delta(a) = 9$
$(s, +)$
0/8
a
c
0/9
1/10
$\delta(S) = \infty$
$(S, +)$
0/10
1/1
s
t
4/9
3/7
b
d
3/3
$\delta(b) = min($ ∞,9-4)=5
$(S, +)$

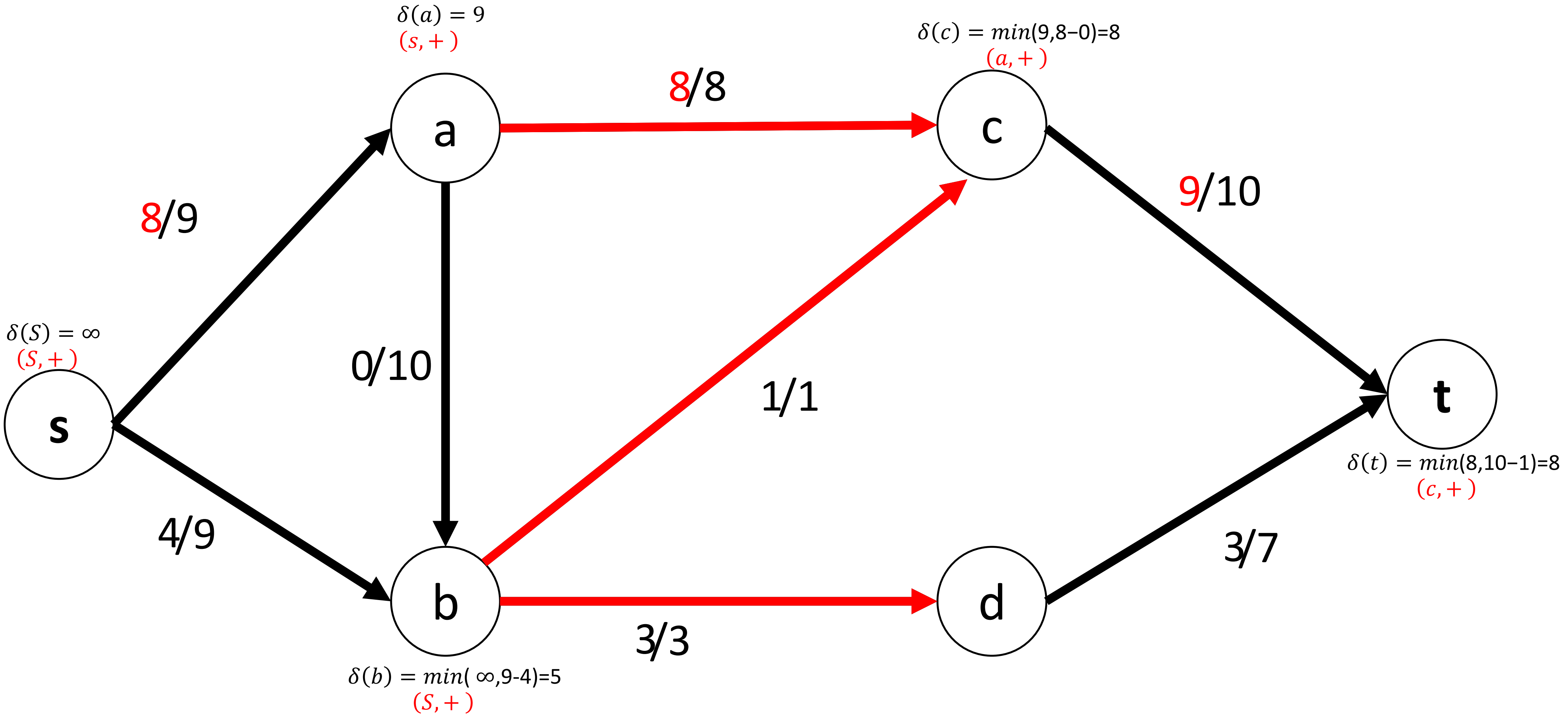
3



δ(a) = 9
(s, +)
δ(c) = min(9,8−0)=8
(a, +)
0/8
a
c
0/9
1/10
δ(S) = ∞
(S, +)
0/10
1/1
s
t
4/9
3/7
b
d
3/3
δ(b) = min(∞,9-4)=5
(S, +)



δ(a) = 9
(s, +)
δ(c) = min(9,8−0)=8
(a, +)
0/8
a
c
0/9
1/10
δ(S) = ∞
(S, +)
0/10
1/1
s
t
δ(t) = min(8,10−1)=8
(c, +)
4/9
3/7
b
d
3/3
δ(b) = min(∞,9-4)=5
(S, +)


δ(a) = 9
(s, +)
δ(c) = min(9,8−0)=8
(a, +)
8/8
a
c
9/10
8/9
δ(S) = ∞
(S, +)
0/10
1/1
s
t
δ(t) = min(8,10−1)=8
(c, +)
4/9
3/7
b
d
3/3
δ(b) = min(∞,9-4)=5
(S, +)

C(f) = 9 + 3 = 12

Definitions

- **Cut :** A cut is a partition of the set of vertex V into two subsets S and S'=V-S. The cut is denoted [S,S'].
- A cut like the set of arcs whose ends are in the different partitions S and S'.
- A cut is called a ***st cut*** if s ∈ S and t ∈ S'.
- An arc (i,j) such that i ∈ S and j ∈ S' is a *forward arc,* and
- An arc (i,j) such that i ∈ S' and j ∈ S is a *backward* arc of the cut [S,S'].
- The ***capacity*** of a cut ***st* cut** (S,S') is the sum of the capacities of the front arcs of this cut.

$$C(\delta(S)) = \sum_{(i,j)\in\delta(S)} C(i,j)$$

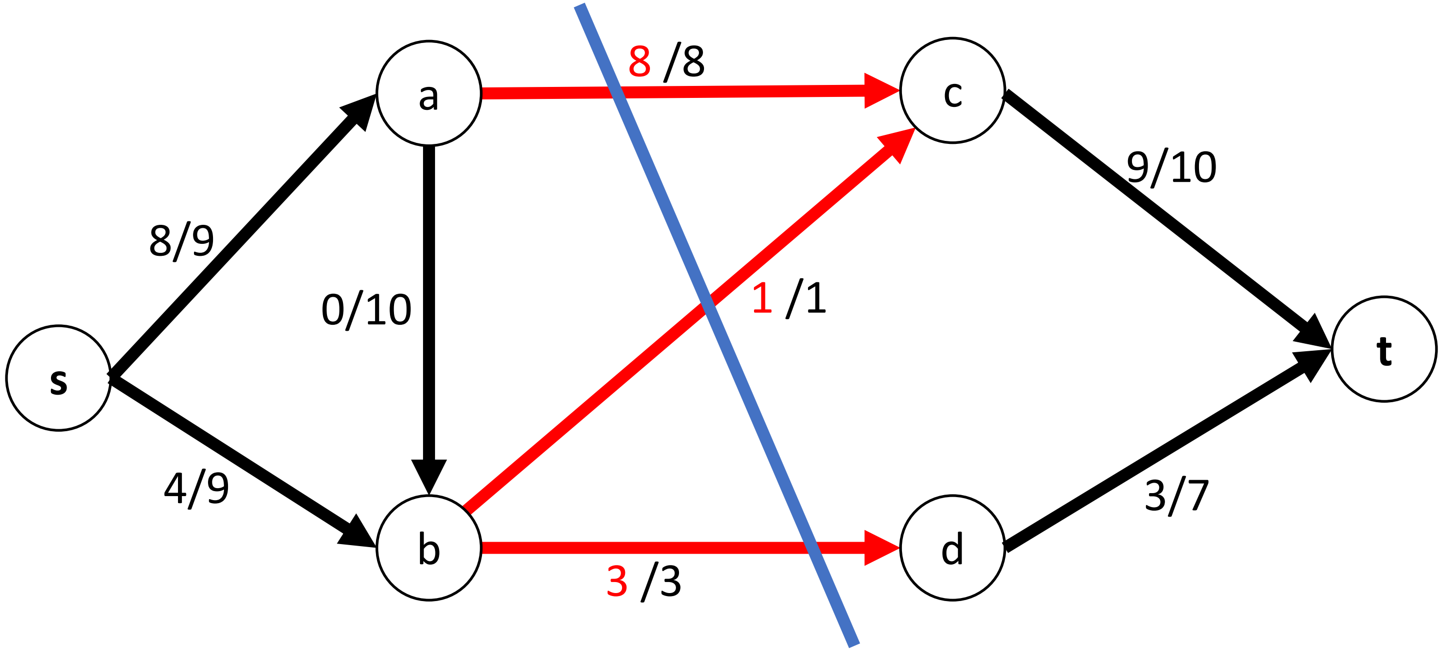
Minimal Cut

- A **st cut** whose capacity is minimum among all *st cuts* is a ***minimum cut*** .
- Any flow from **s** to **t** must pass from **S** to **S'** at some point, and thus uses the capacity of the edges from **S** to **S'**.
- This means that each cut of the graph puts a bound (limit) on the value of the maximum possible flow.

S ={s, a, b} , S'={c, d, t}

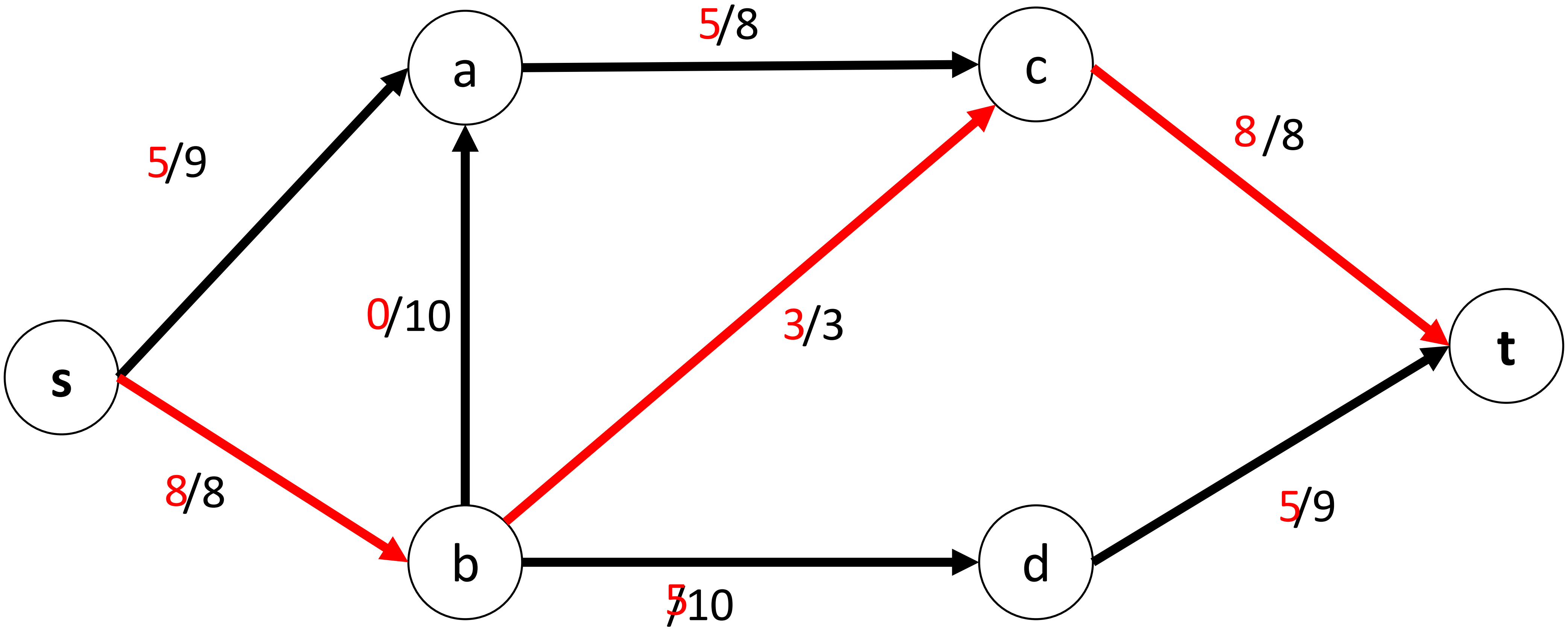
$\delta(S)$ ={(a, c), (b, c), (b, d)}

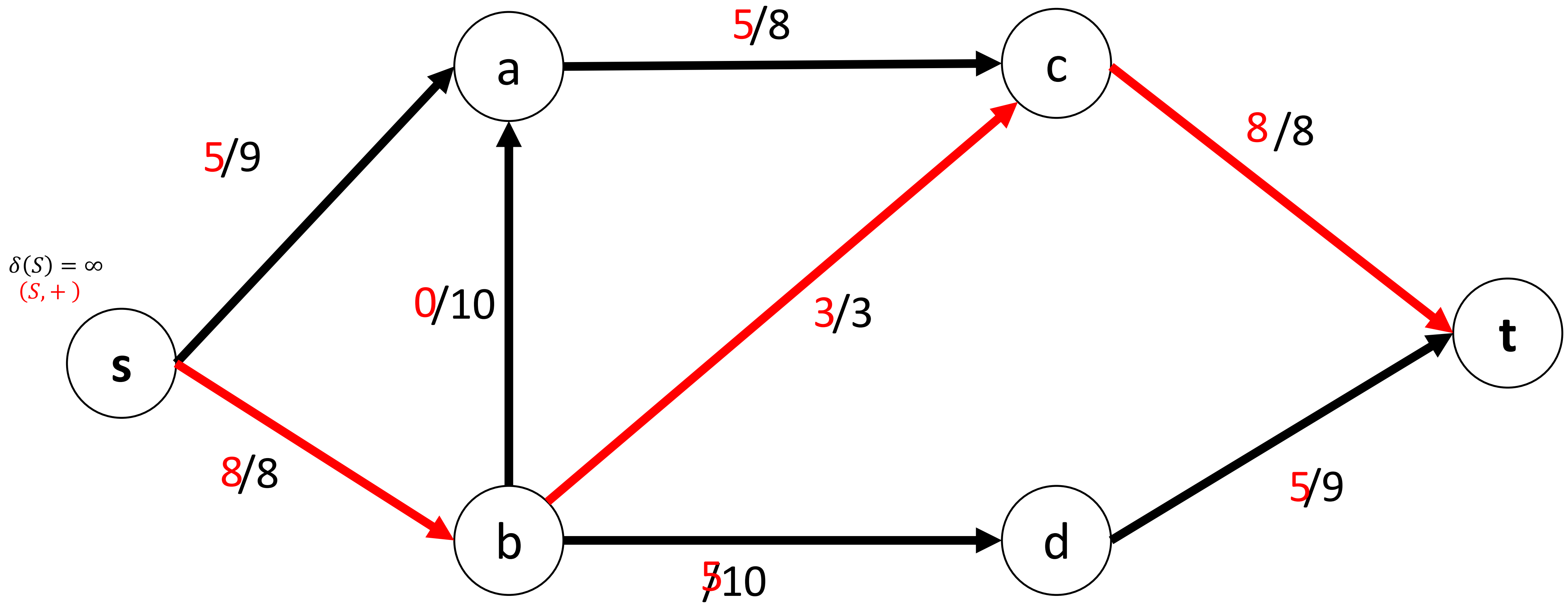
$$C(\delta(S)) = \sum_{(i,j)\in\delta(S)} C(i,j) = 8+1+3 = 12$$



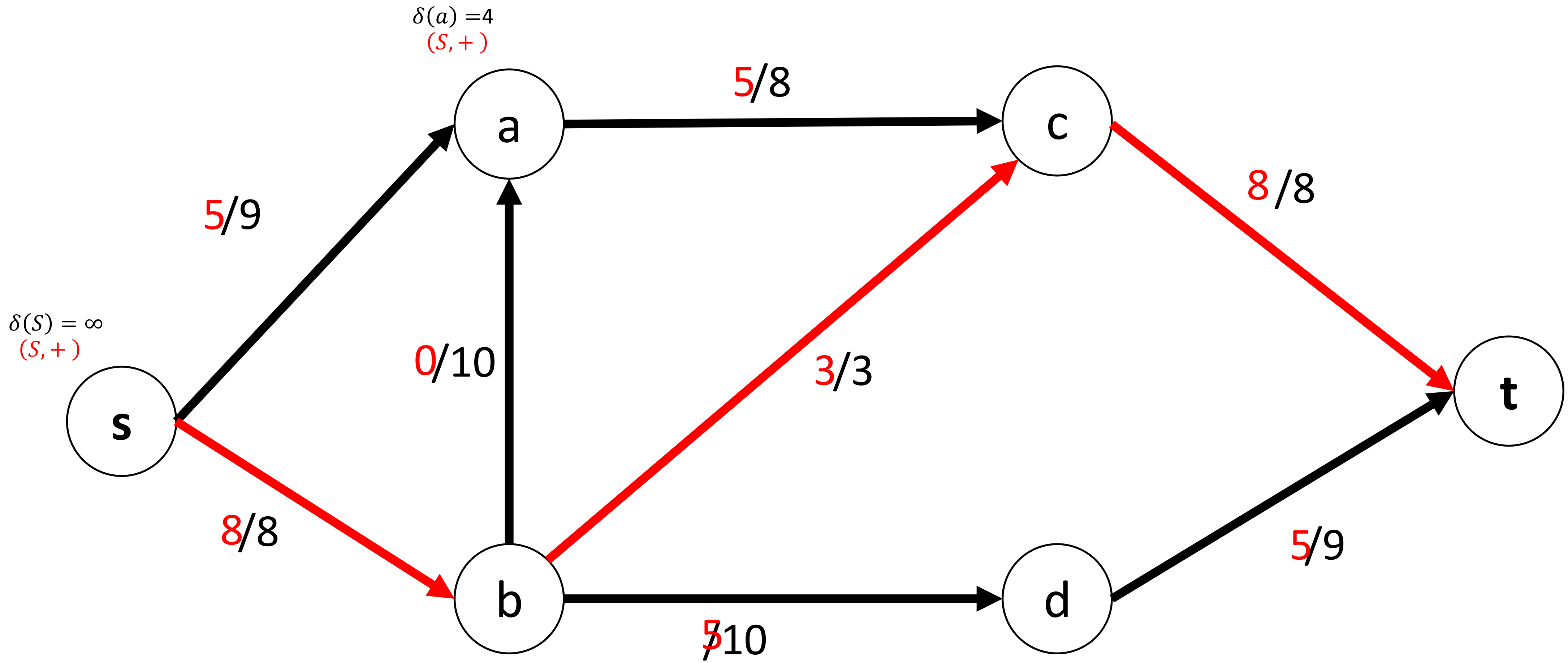
Exercises 1 :

Find the maximum flow by completing the Ford-Fulkerson algorithm .

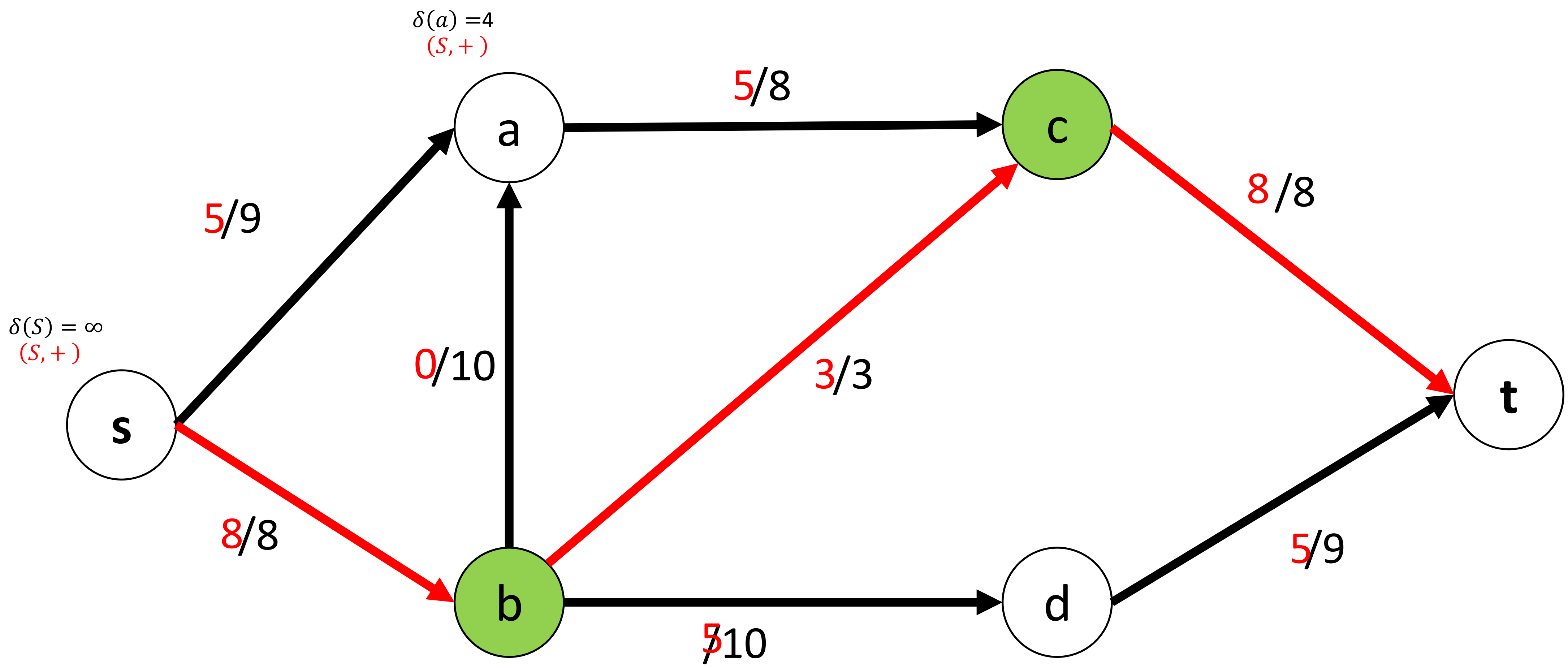




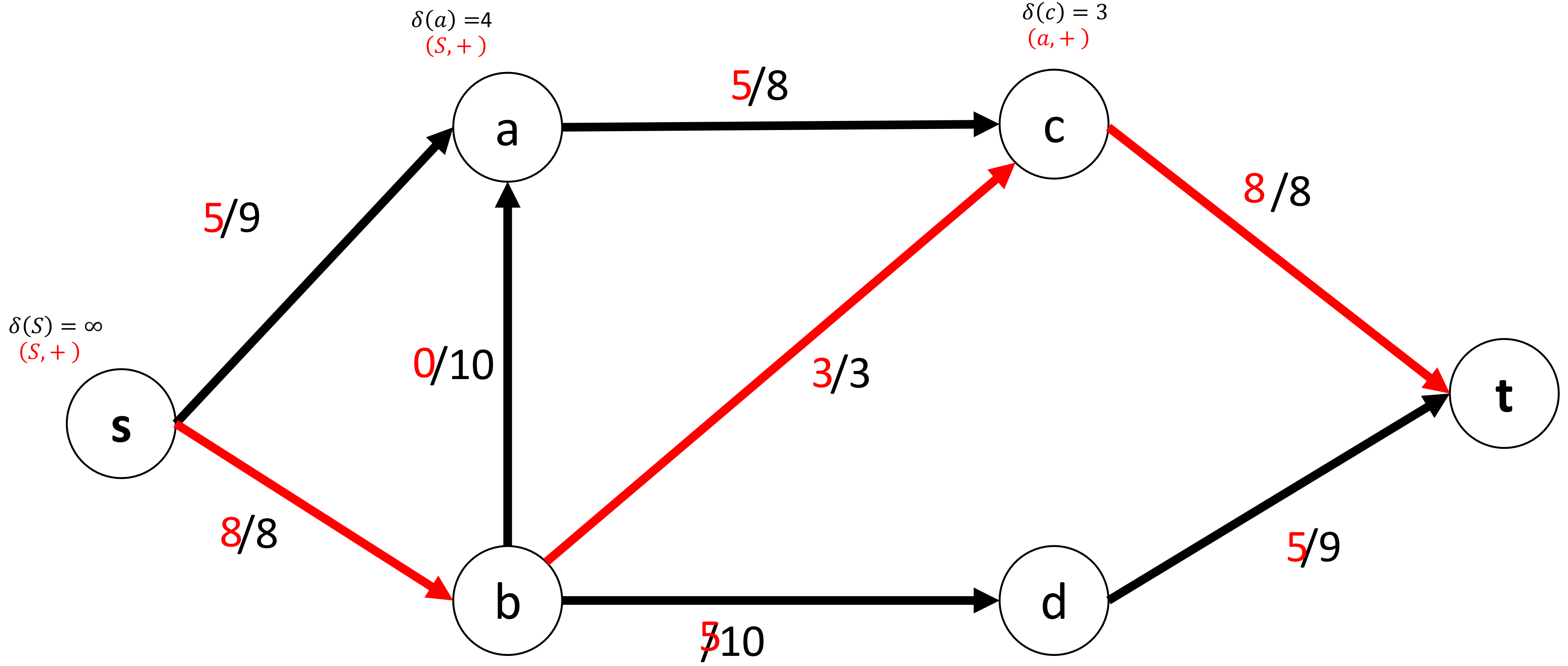
δ(S) = ∞
(S, +)
s
a
b
c
d
t
5/9
8/8
0/10
5/8
3/3
8/8
5/9
5/10



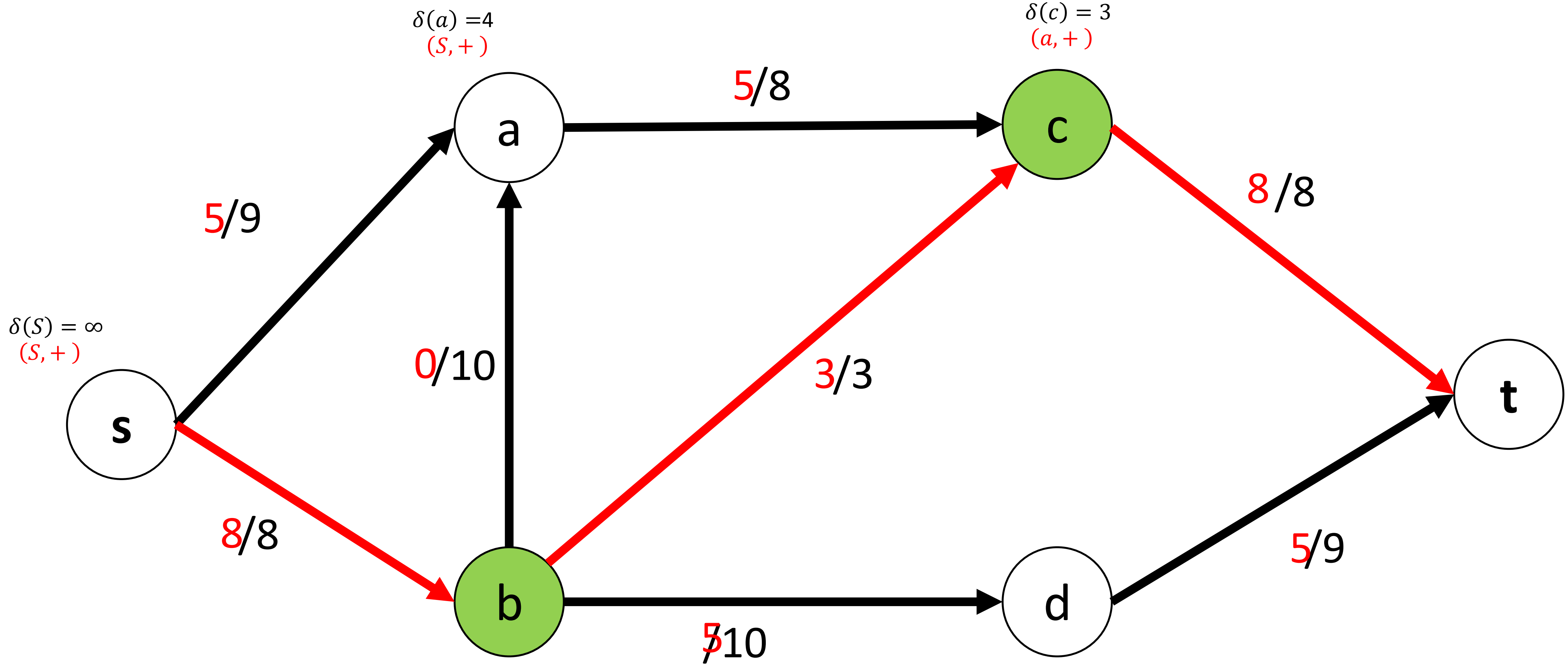
$\delta(a)$ =4
$(S, +)$
5/8
$\delta(S) = \infty$
$(S, +)$
5/9
0/10
8/8
3/3
8/8
5/9
5/10
s
a
b
c
d
t



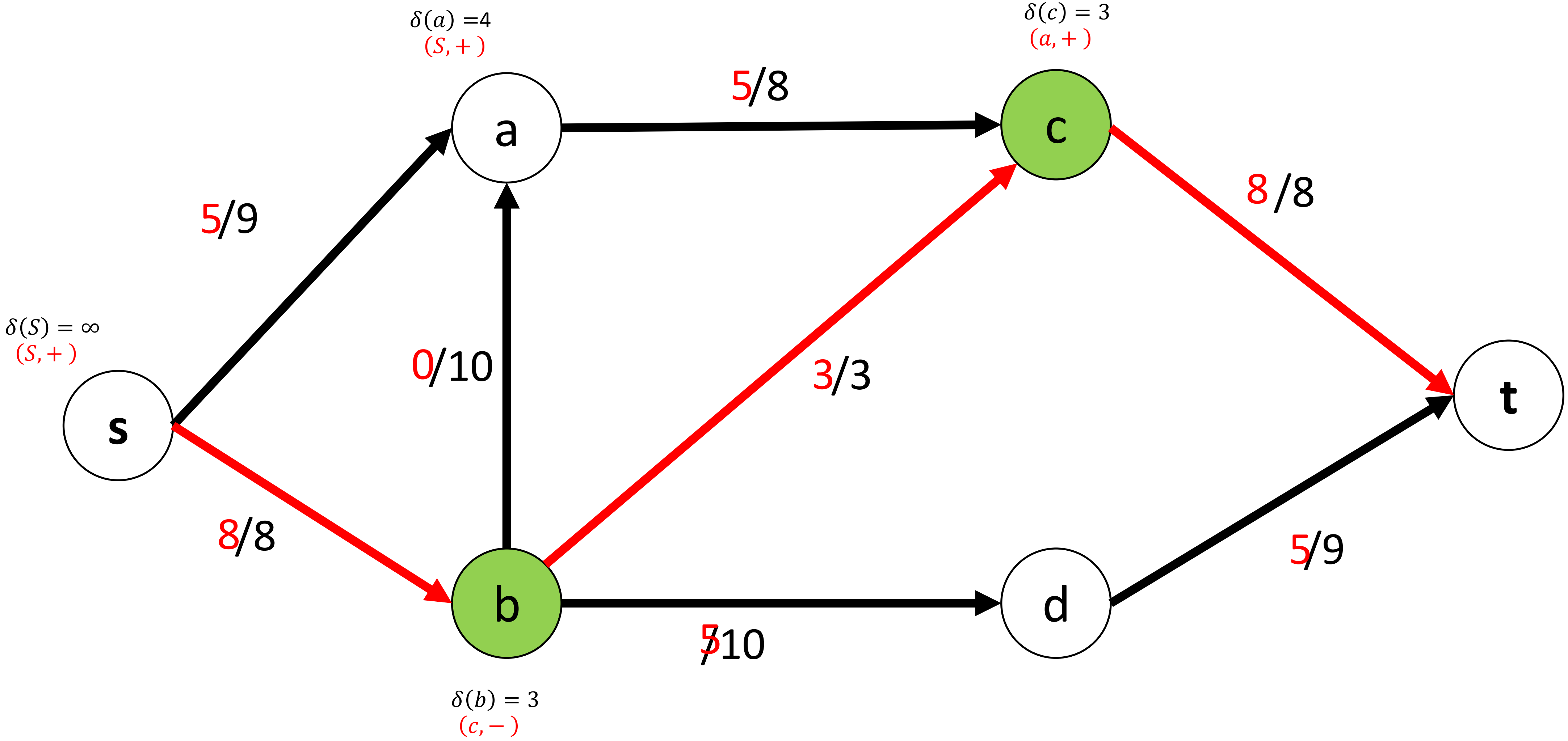
δ(a) =4
(S,+)
δ(S) = ∞
(S,+)
s
a
b
c
d
t
5/9
5/8
8/8
0/10
3/3
8/8
5/10
5/9

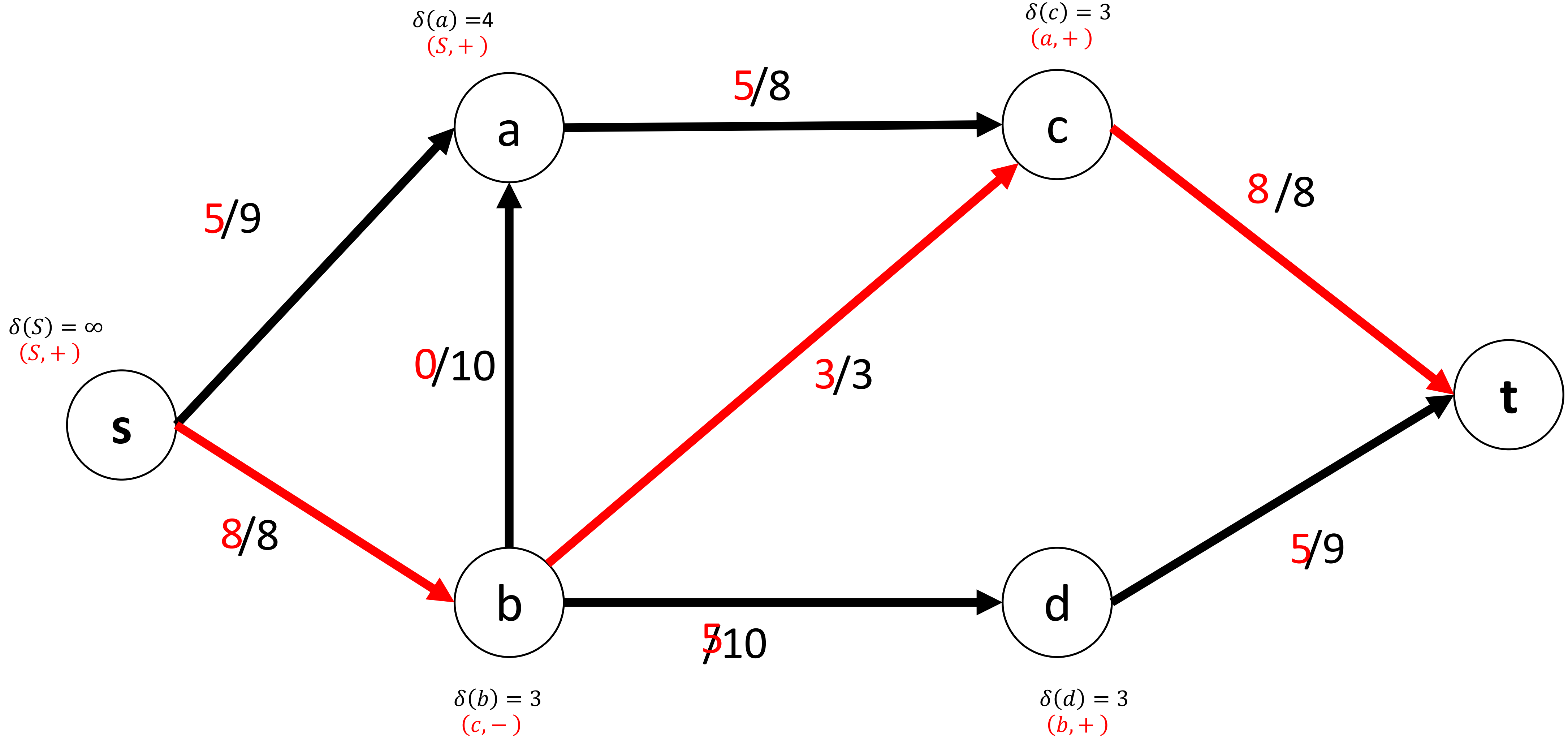


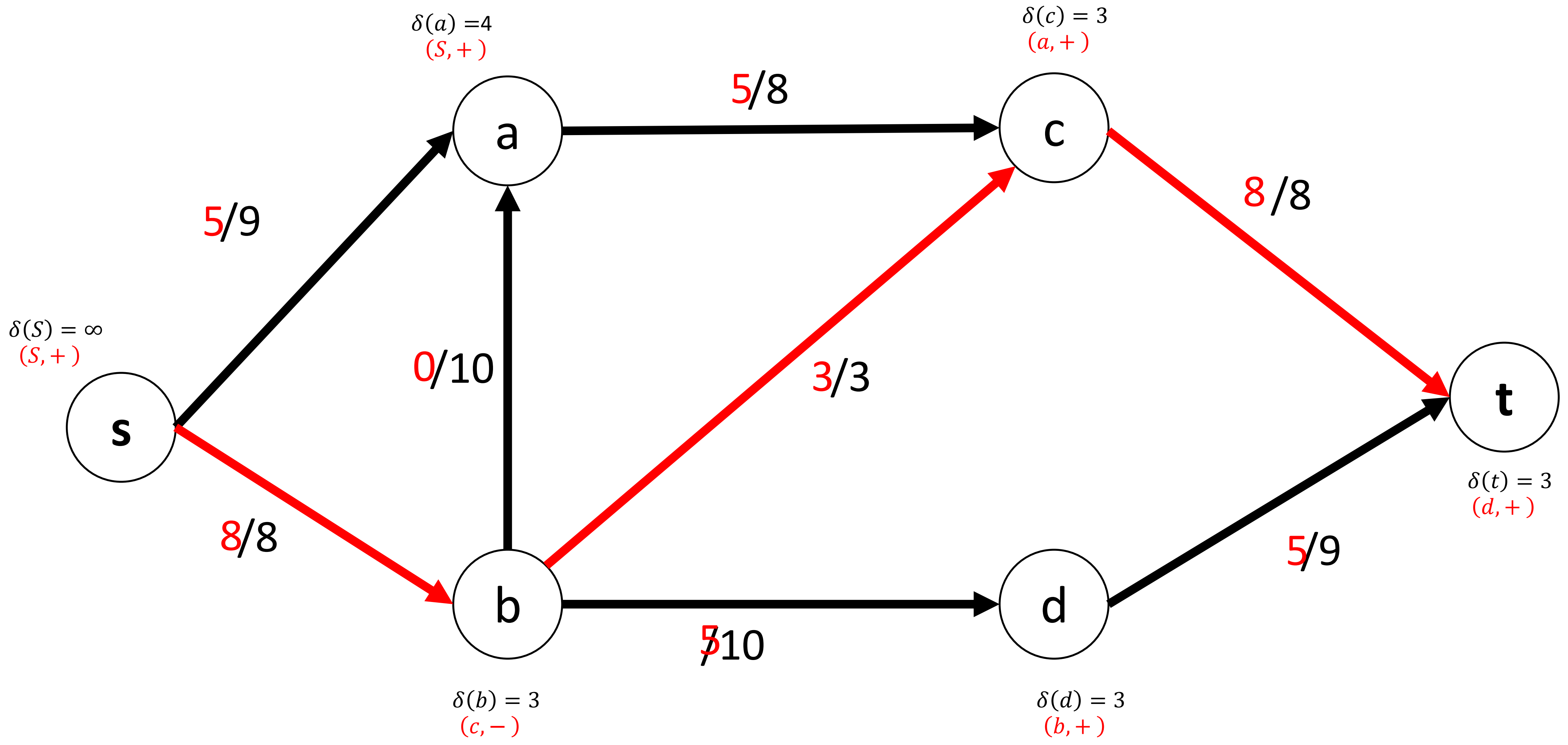
$\delta(a)$ =4
$(S, +)$
$\delta(c) = 3$
$(a, +)$
$\delta(S) = \infty$
$(S, +)$
a
c
s
b
d
t
5/8
5/9
8/8
0/10
3/3
8/8
5/10
5/9



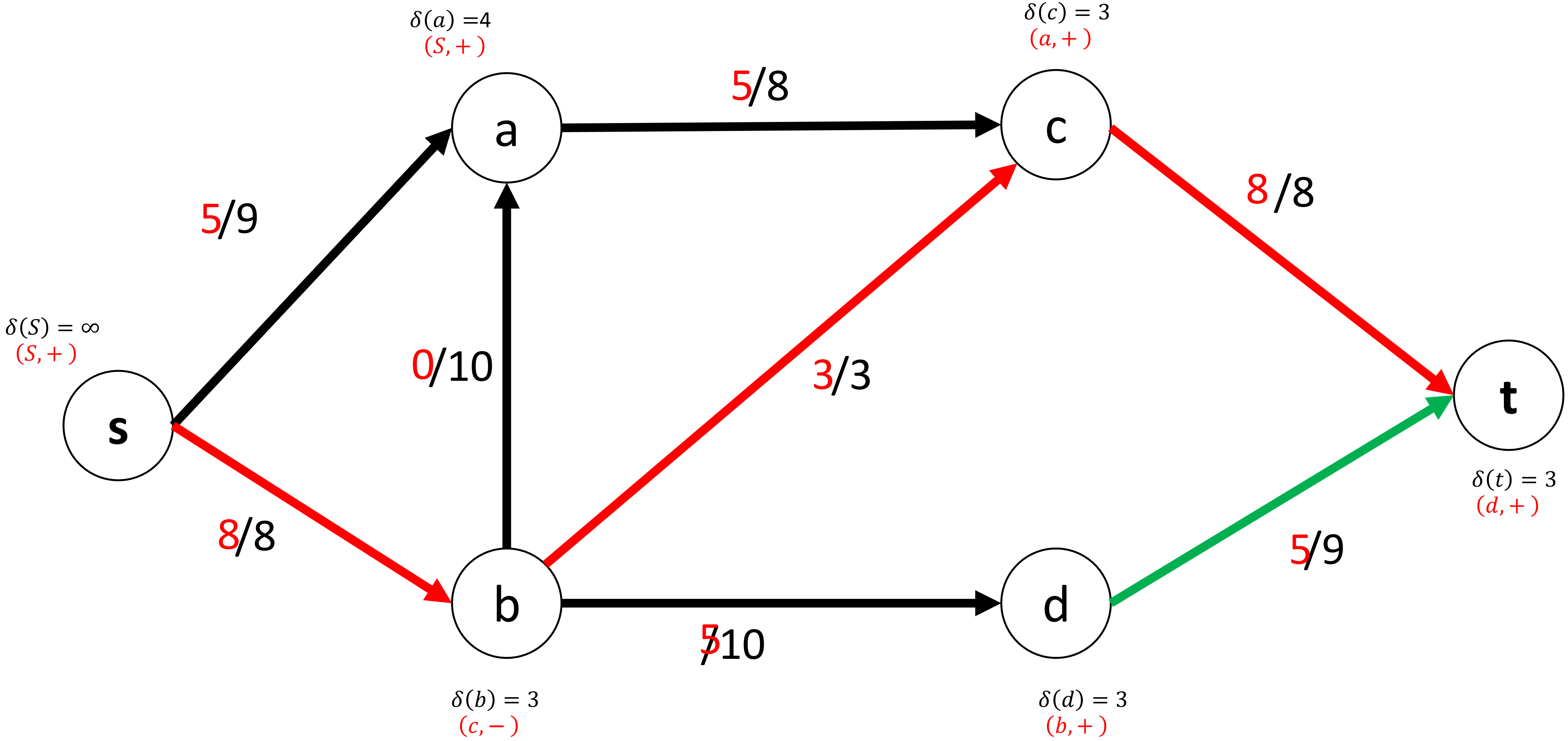
δ(a) =4
(S,+)
δ(c) = 3
(a,+)
δ(S) = ∞
(S,+)
a
c
s
b
d
t
5/8
8 /8
5/9
0/10
3/3
8/8
5/9
5/10


$\delta(a) = 4$
$(S, +)$
$\delta(c) = 3$
$(a, +)$
$\delta(S) = \infty$
$(S, +)$
$\delta(b) = 3$
$(c, -)$
s
a
b
c
d
t
5/9
5/8
0/10
8/8
3/3
8/8
5/10
5/9

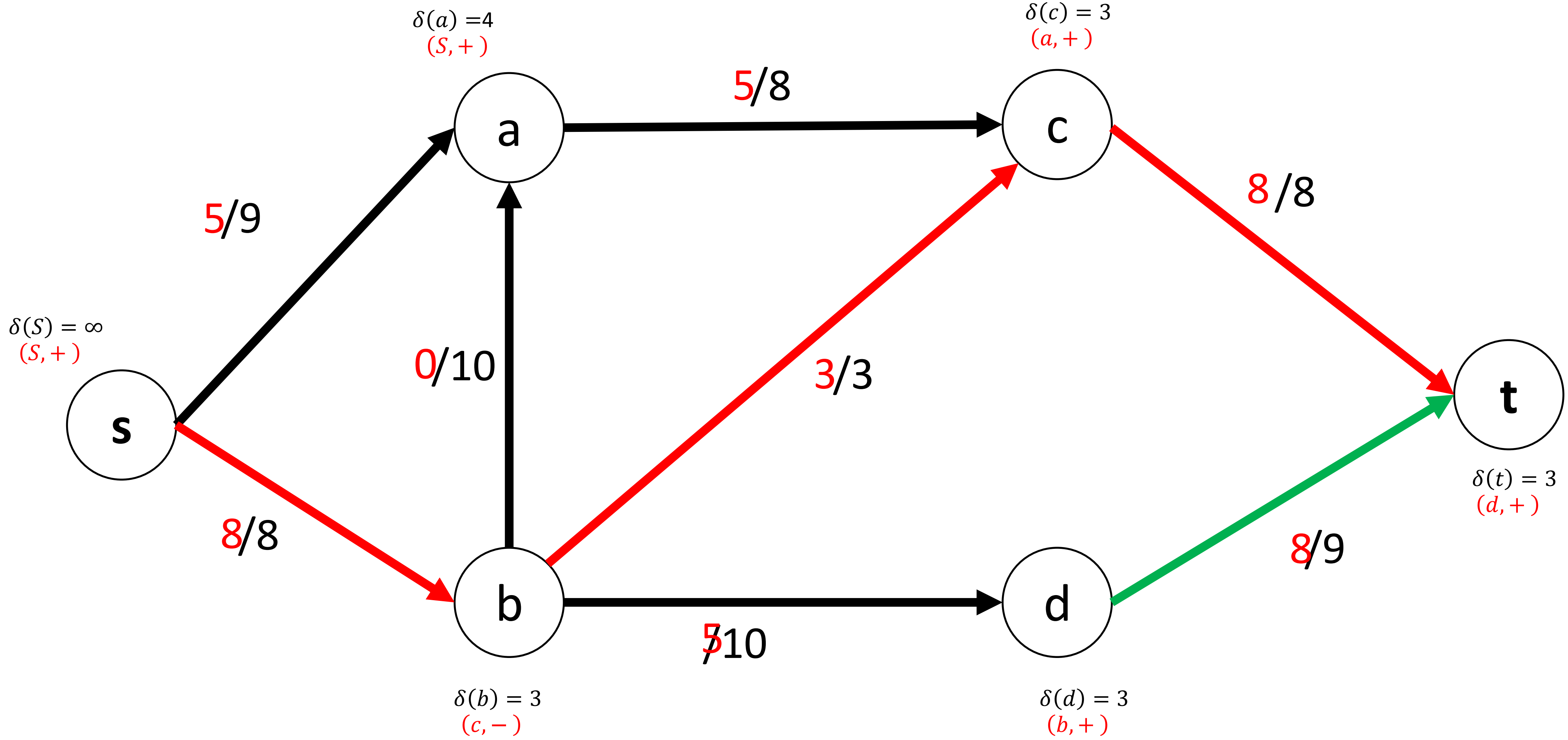

δ(a) =4
(S,+)
δ(c) = 3
(a,+)
5/8
a
c
5/9
8/8
δ(S) = ∞
(S,+)
0/10
3/3
s
t
8/8
5/9
b
d
5/10
δ(b) = 3
(c,−)
δ(d) = 3
(b,+)



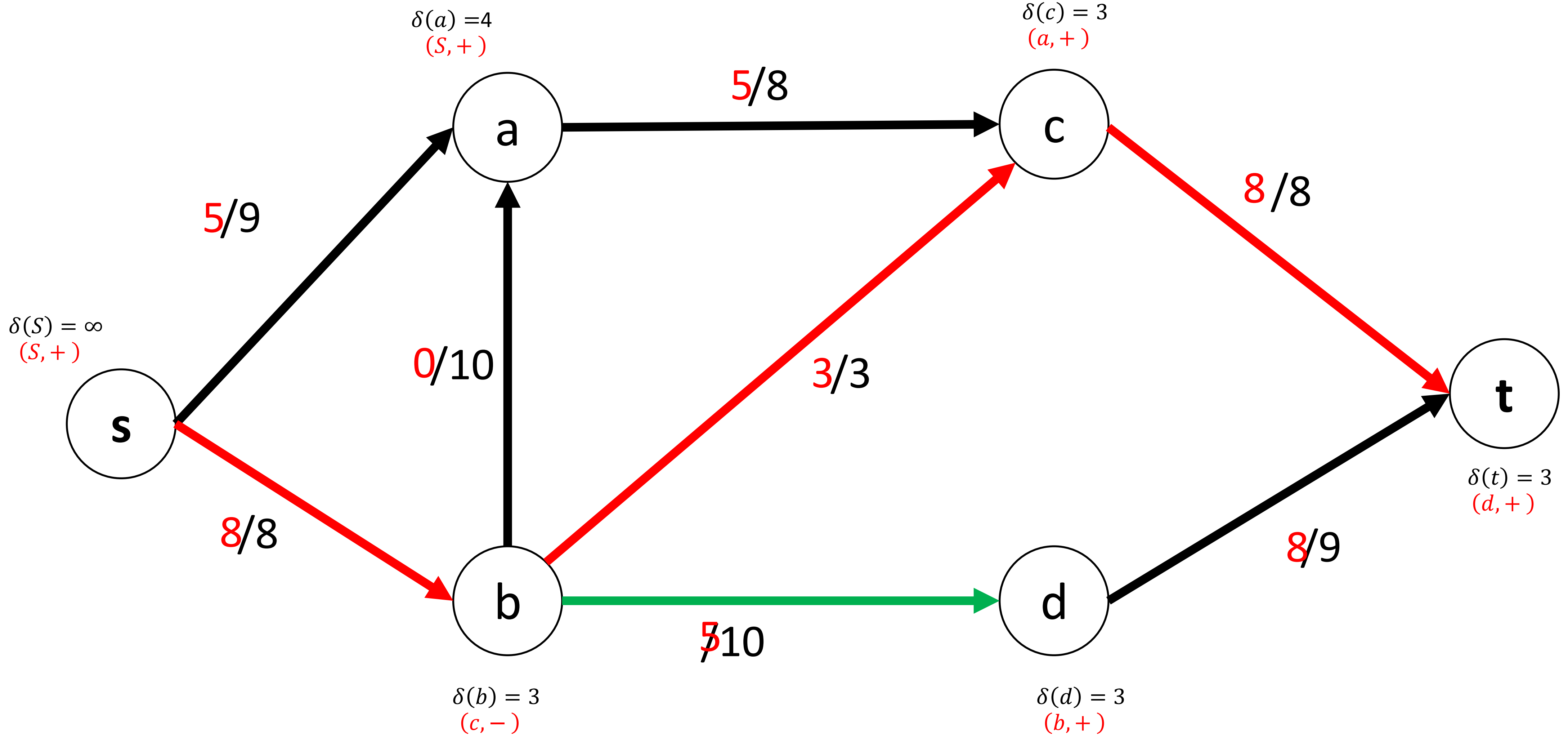
$\delta(a)$ =4
$(S, +)$
$\delta(c) = 3$
$(a, +)$
$\delta(S) = \infty$
$(S, +)$
$\delta(t) = 3$
$(d, +)$
$\delta(b) = 3$
$(c, -)$
$\delta(d) = 3$
$(b, +)$
s
a
b
c
d
t
5/9
8/8
0/10
5/8
3/3
8/8
5/10
5/9



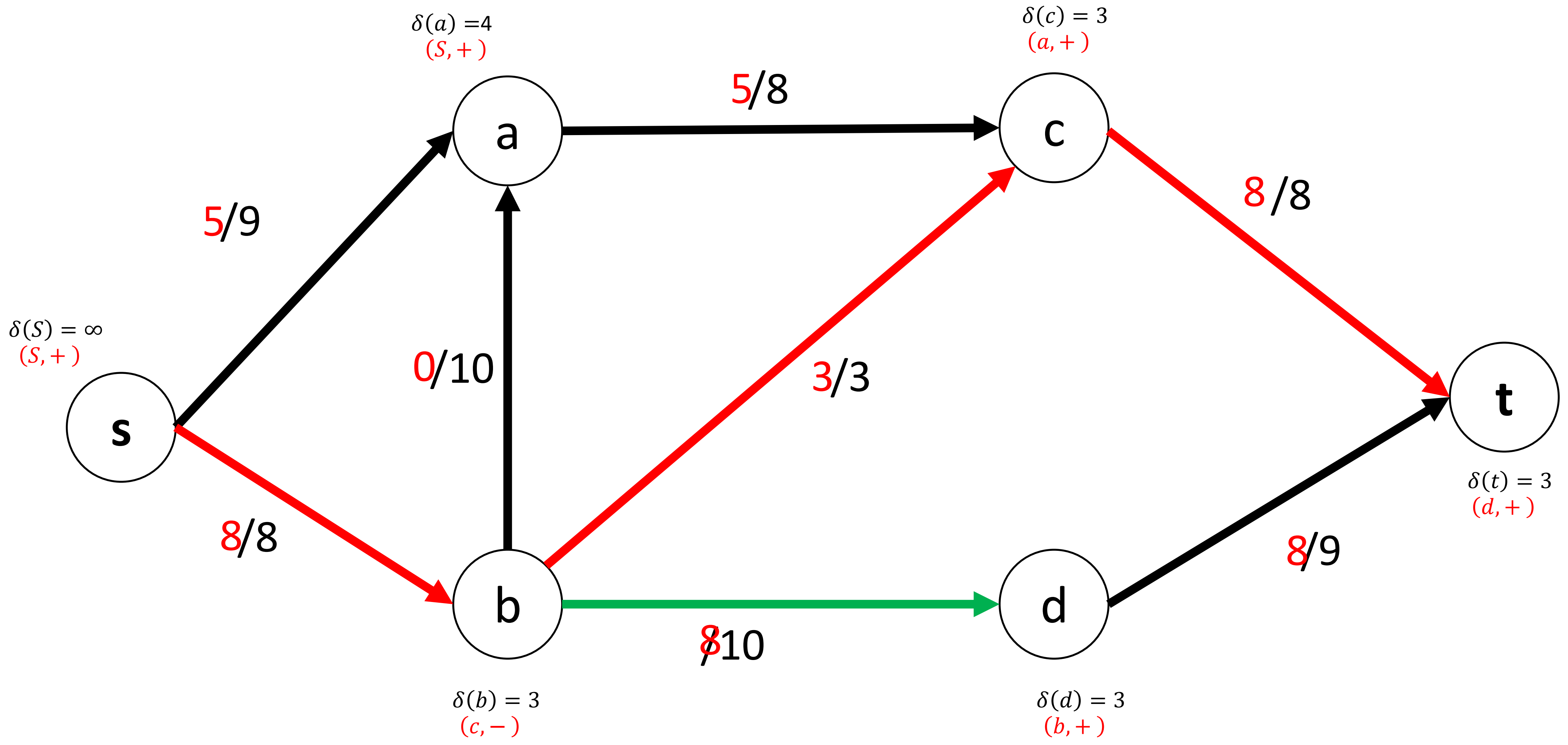
δ(a) =4
(S,+)
δ(c) = 3
(a,+)
5/8
a
c
5/9
8/8
δ(S) = ∞
(S,+)
0/10
3/3
t
s
δ(t) = 3
(d,+)
8/8
5/9
b
d
5/10
δ(b) = 3
(c,−)
δ(d) = 3
(b,+)



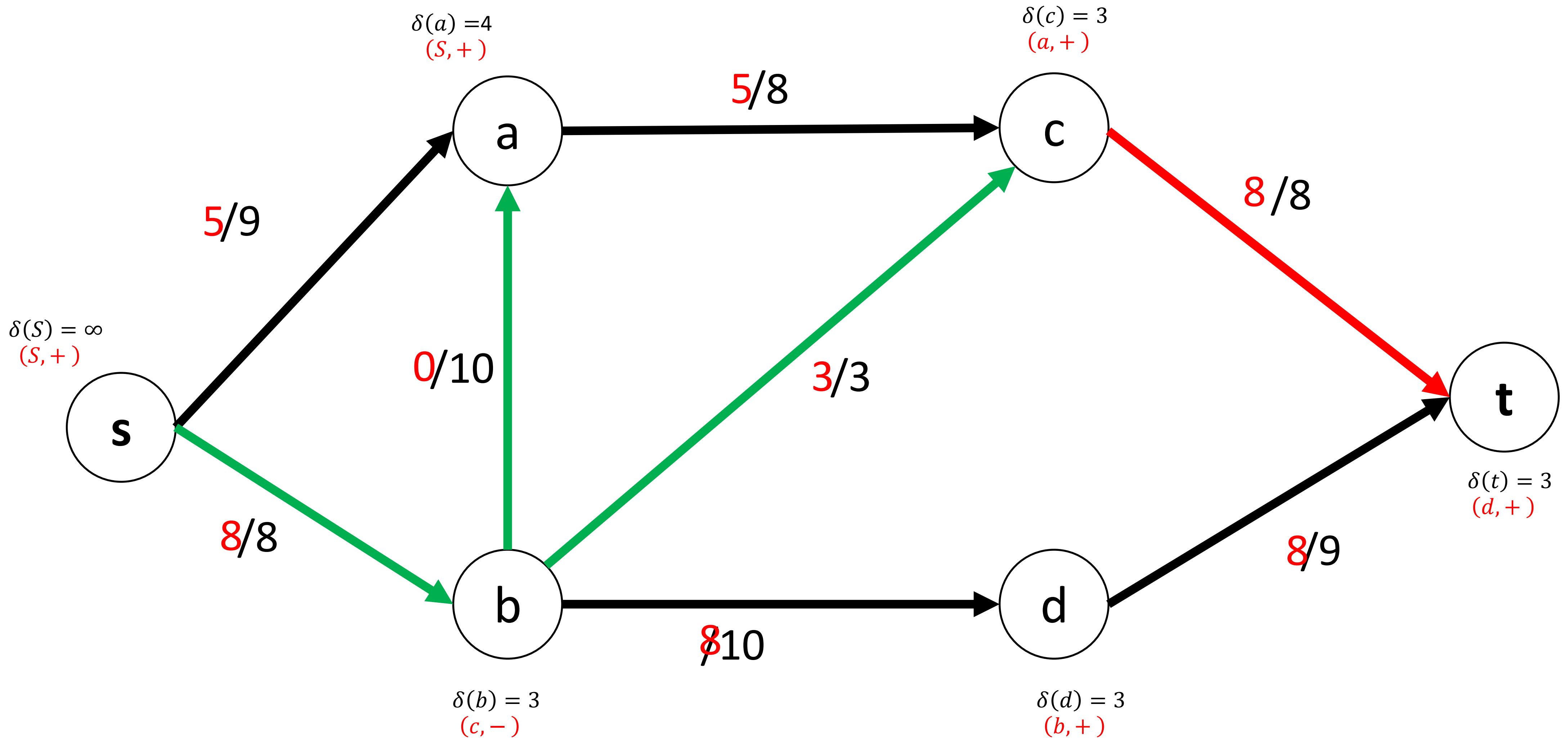
δ(a) =4
(S, +)
5/8
δ(c) = 3
(a, +)
5/9
8 /8
δ(S) = ∞
(S, +)
0/10
3/3
s
a
c
t
b
d
8/8
8/9
δ(t) = 3
(d, +)
5/10
δ(b) = 3
(c, −)
δ(d) = 3
(b, +)



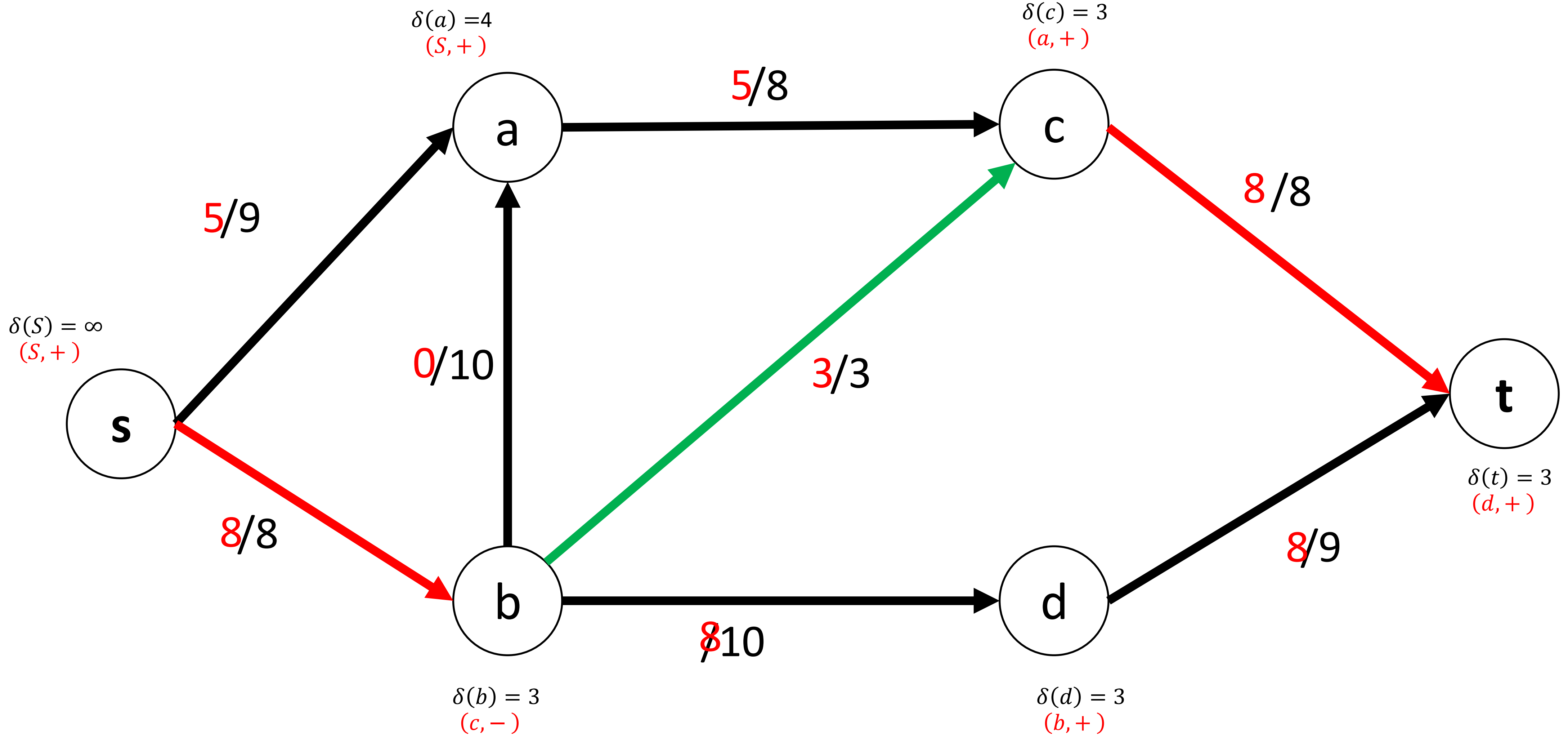
δ(a) =4
(S, +)
5/8
δ(c) = 3
(a, +)
5/9
8 /8
δ(S) = ∞
(S, +)
0/10
3/3
s
a
c
t
b
d
8/8
δ(t) = 3
(d, +)
8/9
5/10
δ(b) = 3
(c, −)
δ(d) = 3
(b, +)



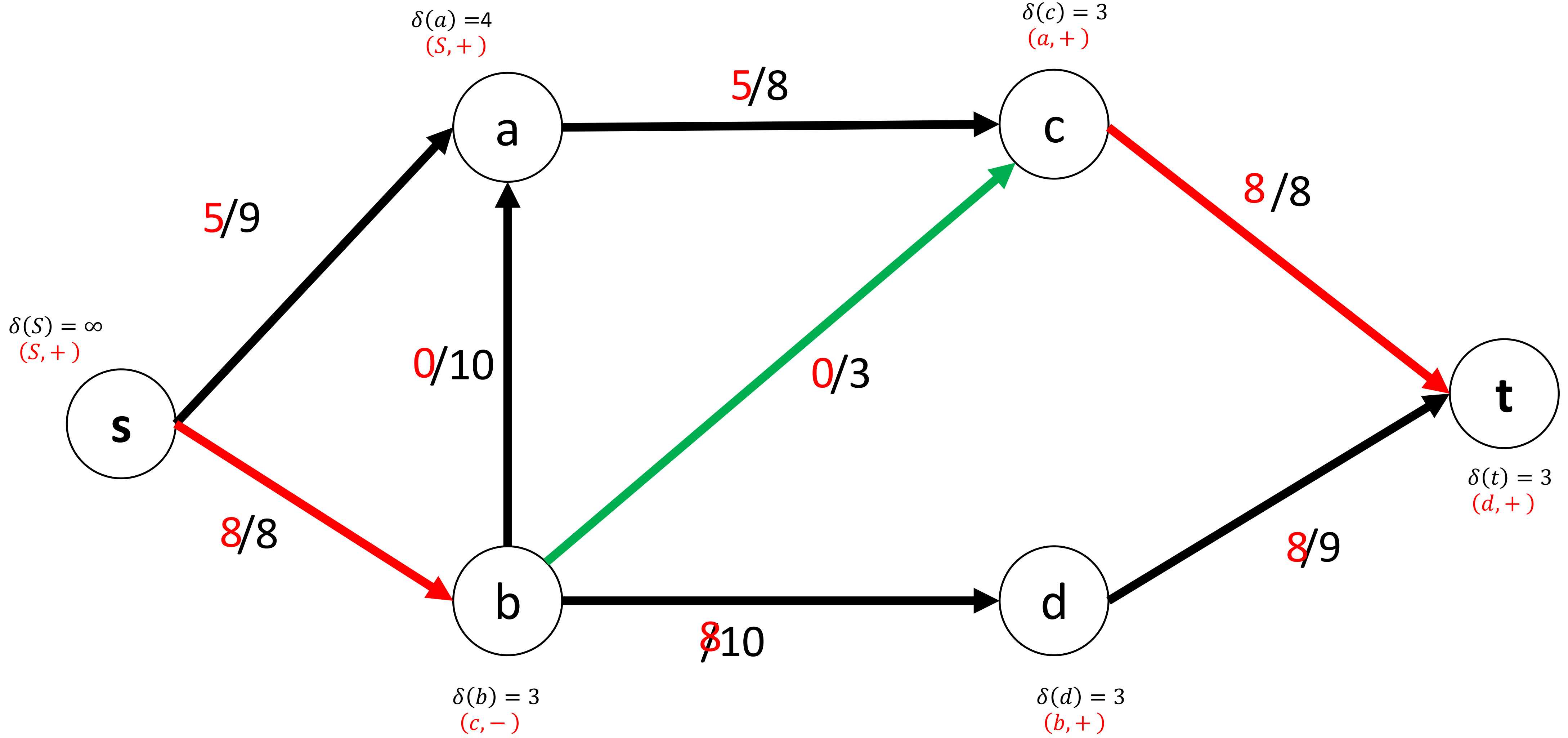
δ(a) =4
(S,+)
δ(c) = 3
(a,+)
δ(S) = ∞
(S,+)
δ(t) = 3
(d,+)
δ(b) = 3
(c,−)
δ(d) = 3
(b,+)
a
c
s
b
d
t
5/9
5/8
8/8
0/10
3/3
8/8
8/9
8/10



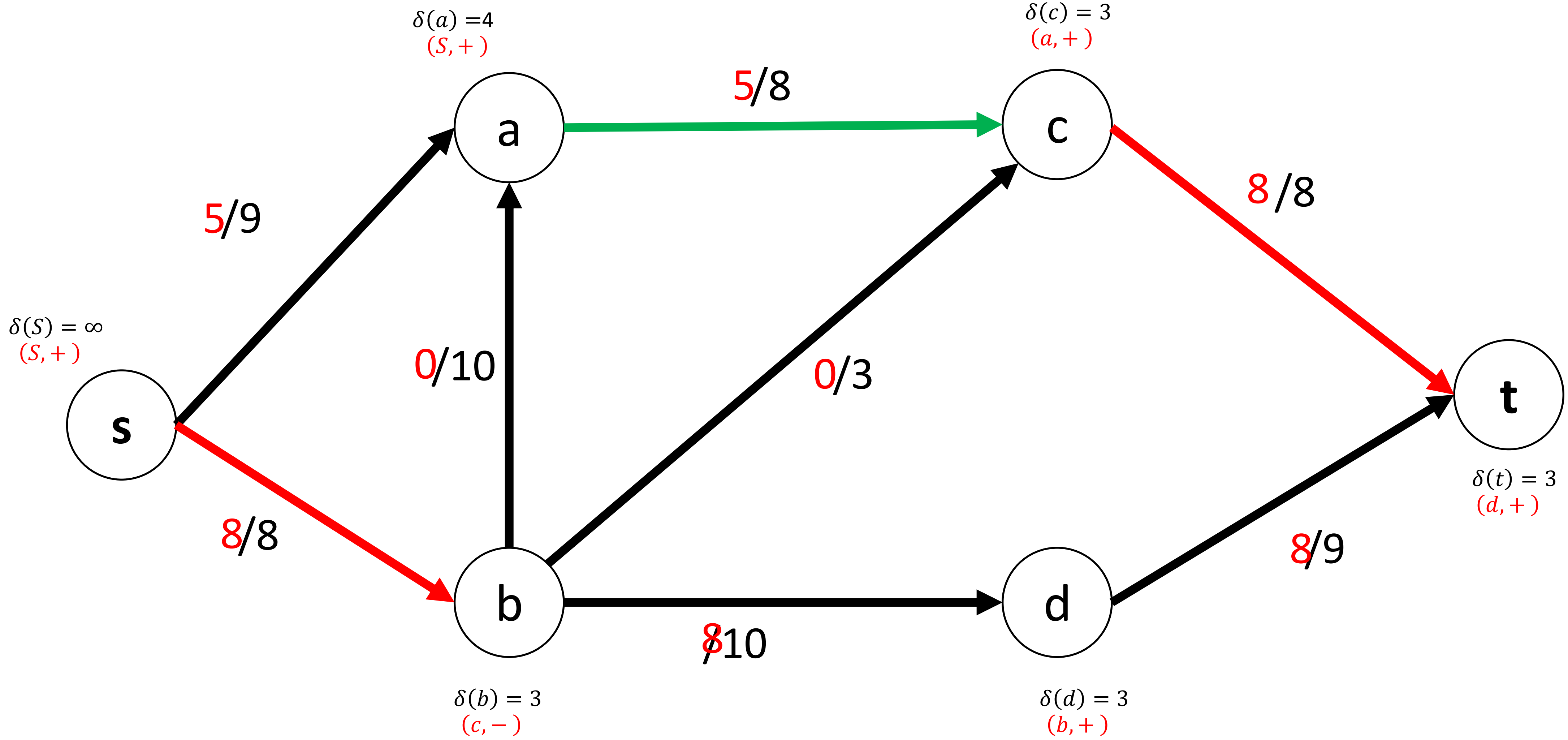
δ(a) =4
(S,+)
5/8
δ(c) = 3
(a,+)
a
c
8/8
5/9
δ(S) = ∞
(S,+)
0/10
3/3
t
s
δ(t) = 3
(d,+)
8/8
8/9
b
d
8/10
δ(b) = 3
(c,−)
δ(d) = 3
(b,+)



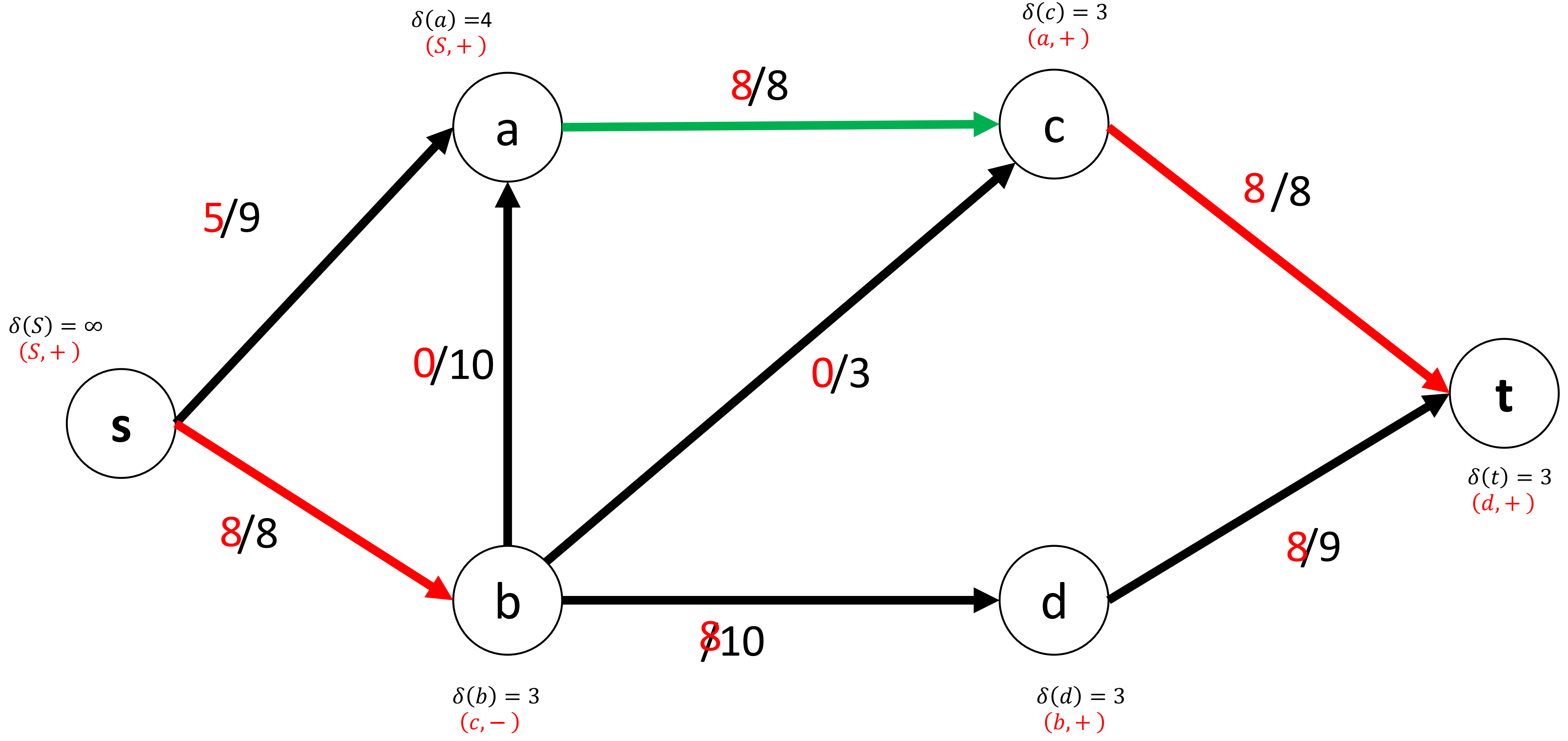
$\delta(a)$ =4
$(S,+)$
$\delta(c) = 3$
$(a,+)$
$\delta(S) = \infty$
$(S,+)$
$\delta(t) = 3$
$(d,+)$
$\delta(b) = 3$
$(c,-)$
$\delta(d) = 3$
$(b,+)$
a
c
s
t
b
d
5/9
5/8
8/8
0/10
3/3
8/8
8/9
8/10



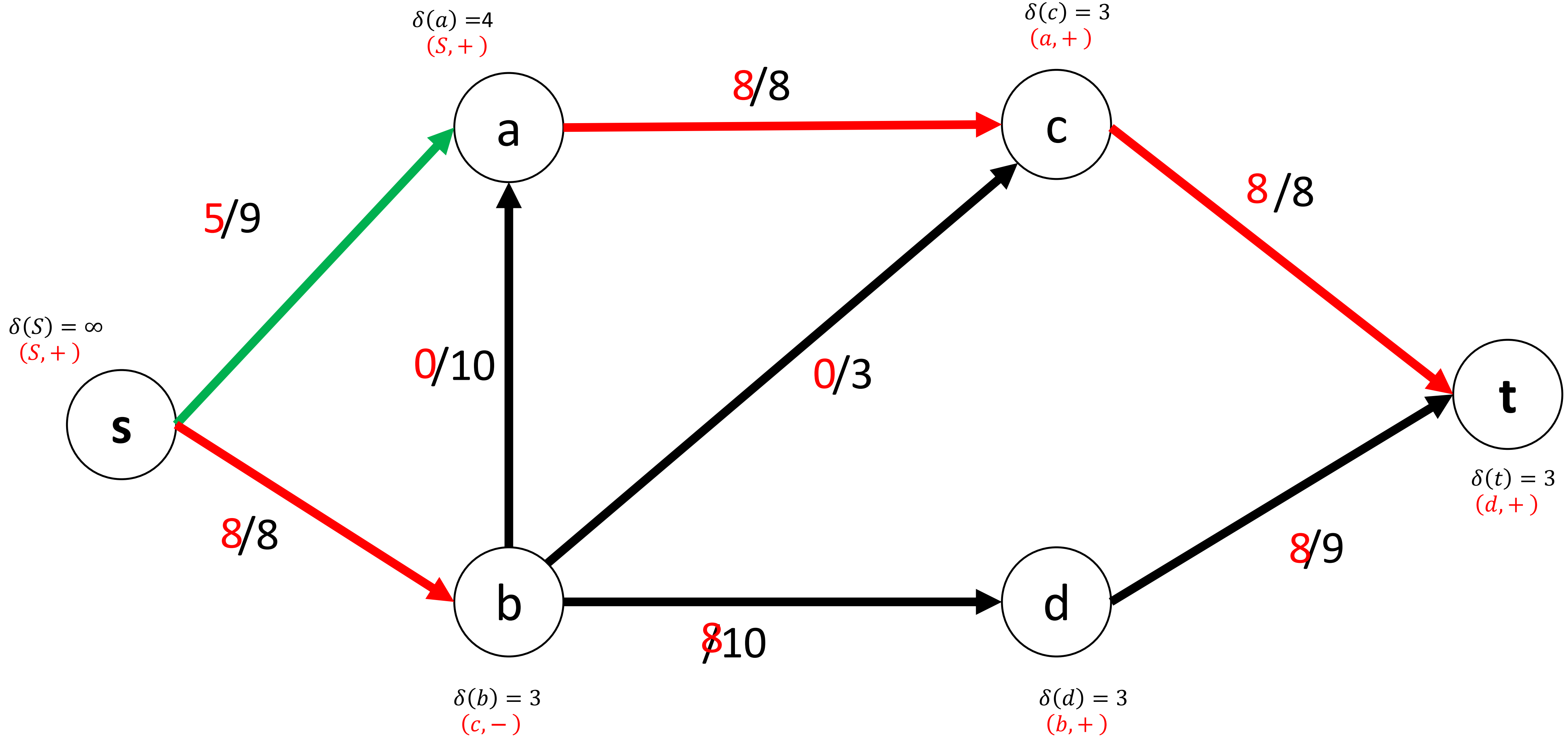
$\delta(a)$ =4
$(S,+)$
$\delta(c) = 3$
$(a,+)$
$\delta(S) = \infty$
$(S,+)$
$\delta(t) = 3$
$(d,+)$
$\delta(b) = 3$
$(c,-)$
$\delta(d) = 3$
$(b,+)$
s
a
b
c
d
t
5/9
8/8
5/8
0/10
0/3
8/8
8/10
8/9



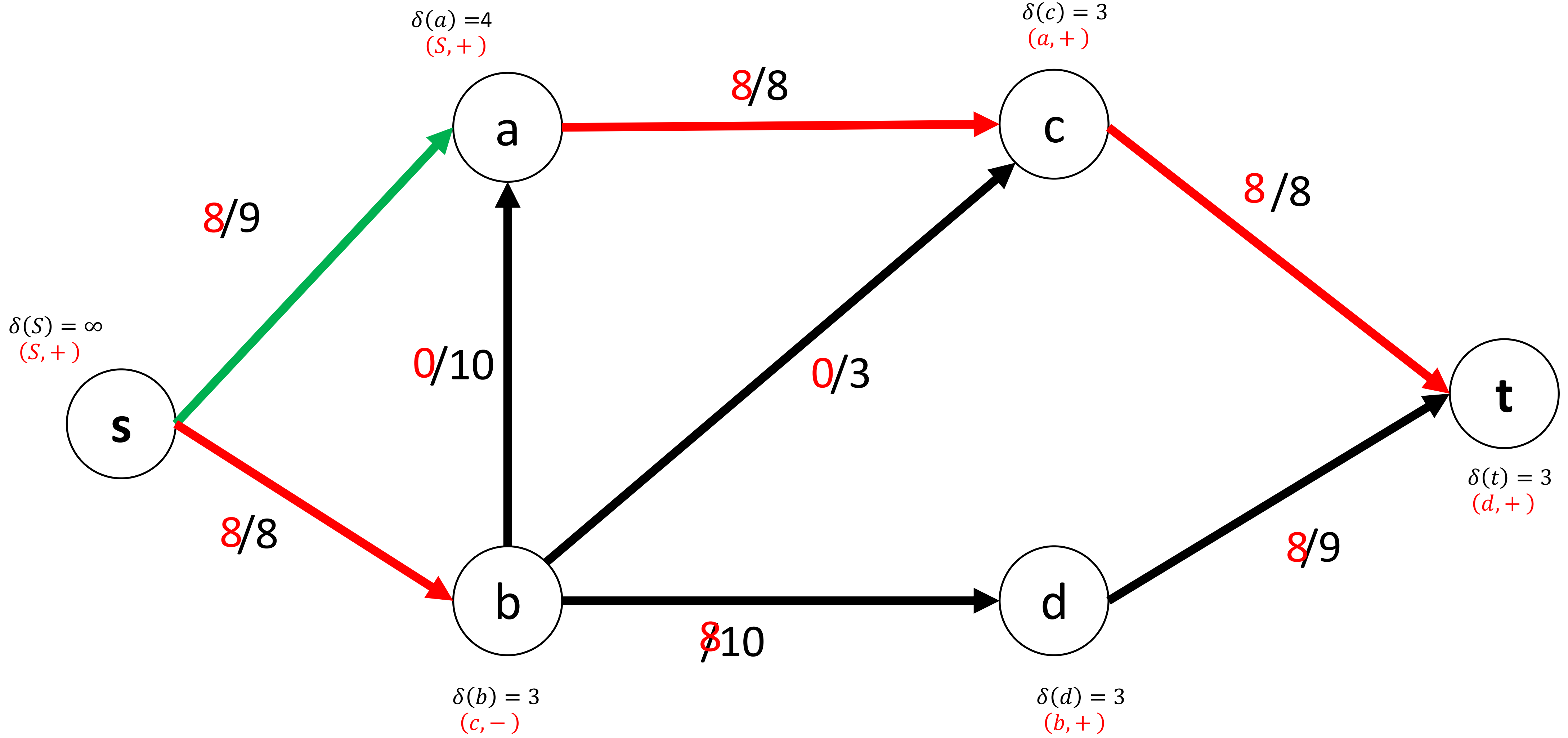
δ(a) =4
(S,+)
5/8
δ(c) = 3
(a,+)
a
c
5/9
8 /8
δ(S) = ∞
(S,+)
0/10
0/3
s
t
δ(t) = 3
(d,+)
8/8
8/9
b
d
8/10
δ(b) = 3
(c,−)
δ(d) = 3
(b,+)



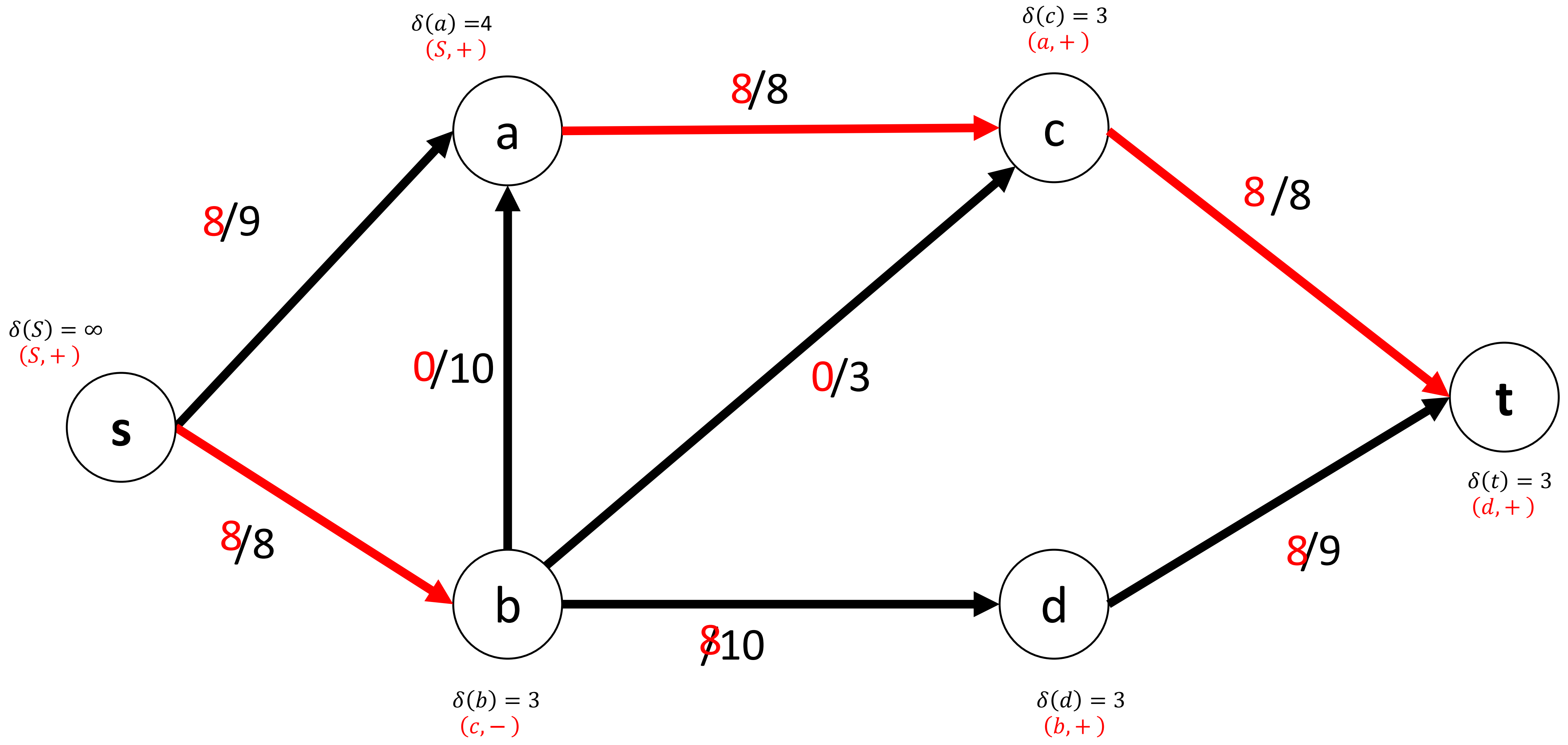
δ(a) =4
(S,+)
δ(c) = 3
(a,+)
8/8
a
c
5/9
8/8
δ(S) = ∞
(S,+)
0/10
0/3
s
t
δ(t) = 3
(d,+)
8/8
8/9
b
d
8/10
δ(b) = 3
(c,−)
δ(d) = 3
(b,+)



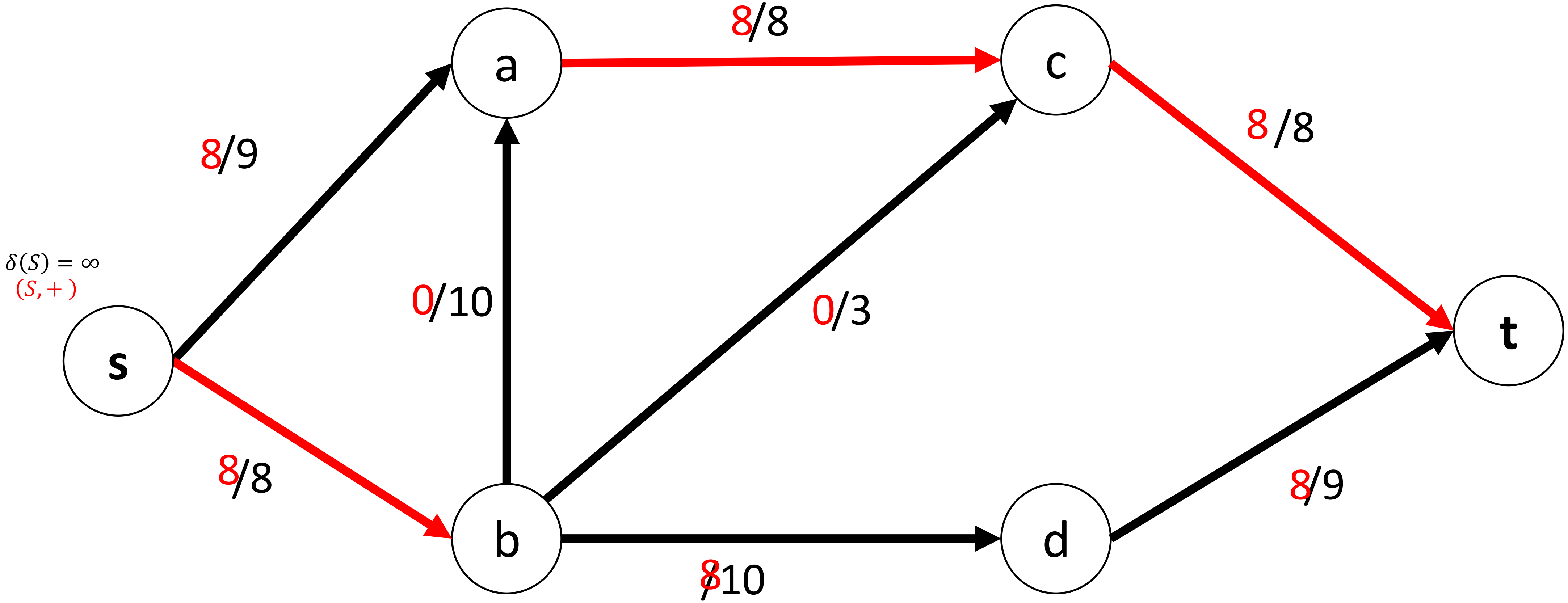
$\delta(S) = \infty$
$(S, +)$
s
$\delta(a) = 4$
$(S, +)$
a
$\delta(c) = 3$
$(a, +)$
c
t
$\delta(t) = 3$
$(d, +)$
b
$\delta(b) = 3$
$(c, -)$
d
$\delta(d) = 3$
$(b, +)$
5/9
8/8
0/10
8/8
0/3
8/8
8/10
8/9

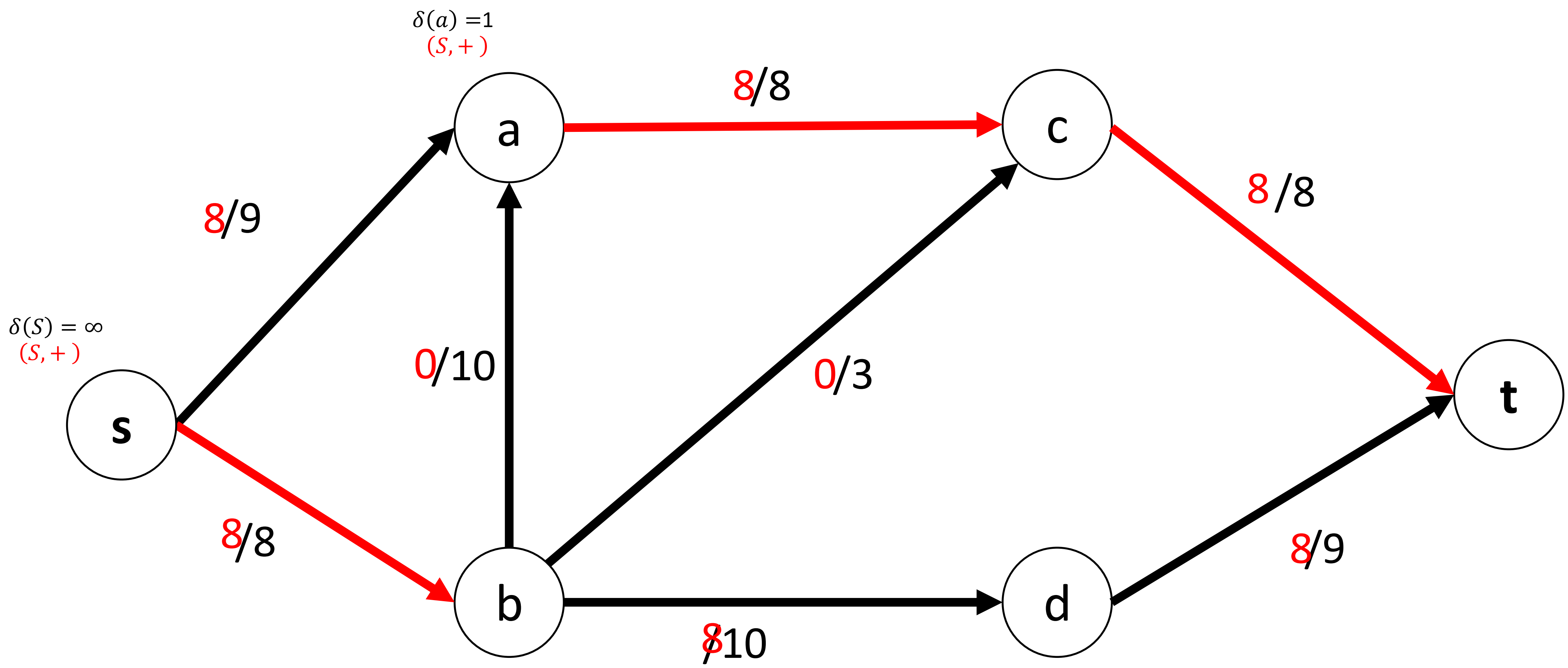


δ(a) =4
(S,+)
δ(c) = 3
(a,+)
8/8
a
c
8/9
8/8
δ(S) = ∞
(S,+)
0/10
0/3
s
t
δ(t) = 3
(d,+)
8/8
8/9
b
d
8/10
δ(b) = 3
(c,−)
δ(d) = 3
(b,+)

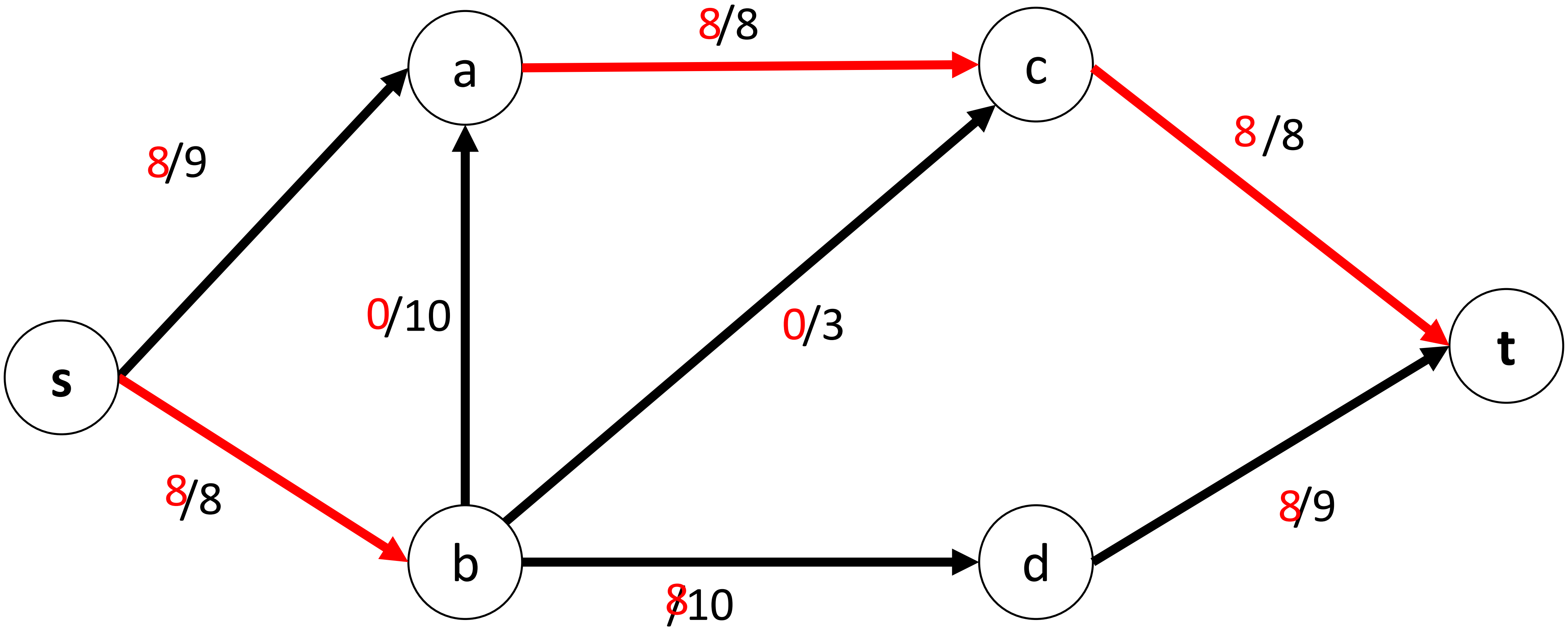


δ(a) =4
(S,+)
δ(c) = 3
(a,+)
8/8
a
c
8/9
8/8
δ(S) = ∞
(S,+)
0/10
0/3
t
s
δ(t) = 3
(d,+)
8/8
8/9
b
d
8/10
δ(b) = 3
(c,−)
δ(d) = 3
(b,+)

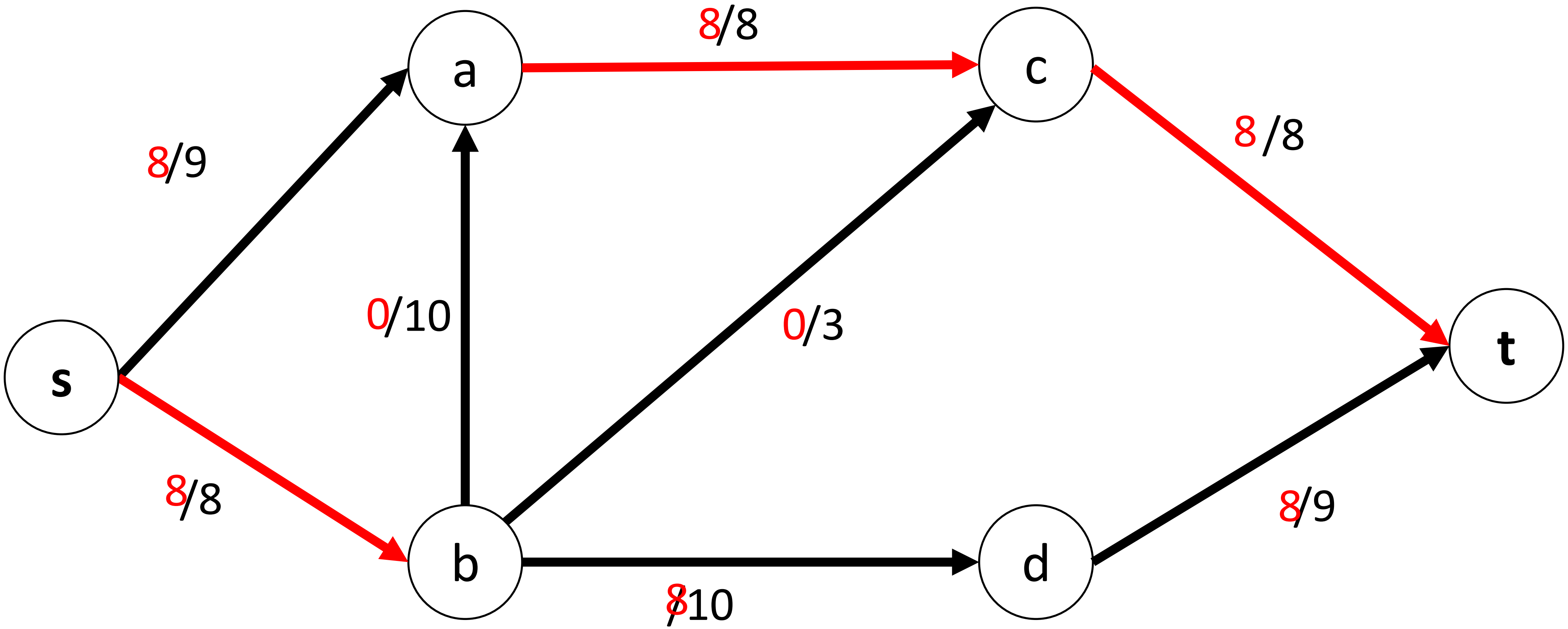

δ(S) = ∞
(S, +)
s
a
b
c
d
t
8/9
8/8
0/10
8/8
0/3
8/8
8/10
8/9



$\delta(a)$ =1
$(S, +)$
$\delta(S) = \infty$
$(S, +)$
a
c
s
t
b
d
8/9
8/8
8/8
0/10
0/3
8/8
8/9
8/10



C(f) = 8 + 8 = 1 6



$$C(f) = 8 + 8 = 16$$

S ={s, a} , S'={ b, c, d, t}

$\delta(S)$ ={(a, c), (S, b)}

$C\big(\delta(S)\big) = \sum_{(i,j)\in\delta(S)} C(i,j)$ = 8+8 = 16